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Pointwise Estimates on the Green's Function for a Scalar Linear Convection–Diffusion Equation

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Pointwise estimates are found on Green's functions for the scalar linear convection–diffusion equations that arise when a scalar conservation law with non-constant diffusion is linearized about a viscous shock profile of arbitrary strength. The estimates take the form of Gaussian kernels centered around paths determined by the (typically different) asymptotic states of the convection function. The analysis extends the spectral transform method to the non-constant coefficient case. © 1999 Academic Press

1. INTRODUCTION

In this paper we obtain pointwise estimates on Green's functions for scalar linear convection–diffusion equations of the forms

$$v_t + (a(x)\, v)_x = (b(x)\, v_x)_x, \qquad u, x, t \in \mathbb{R}, \quad t > 0 \tag{1.1}$$

and

$$v_t + a(x)\, v_x = b(x)\, v_{xx}, \qquad u, x, t \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

with $a(x), b(x) \in C^K(\mathbb{R})$ for some $K \geqslant 1$, $a(x)$, $b(x)$ asymptotically constant at $x = \pm\infty$, and $b(x) \geqslant b_0 > 0$.

Our analysis is motivated by the study of nonlinear stability of viscous shock waves. Equation (1.1) is precisely the form of equation that arises from linearizing the conservation law

$$u_t + f(u)_x = (b(u)\, u_x)_x \tag{1.3}$$

about a stationary scalar viscous shock profile. Also, for weak shocks in systems, equations of form (1.1) approximately govern each characteristic field [G, L1, SX]. Equations of form (1.2) arise when (1.1) is written in terms of the integrated variable $V(x, t) = \int_0^x v(\xi, t)\, d\xi$ [G]. Additionally, the adjoint of (1.1) can be put into form (1.2) simply by rearranging terms.

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We obtain *pointwise* estimates on the Green's function for (1.1) because in [L2] Liu has shown the need for such estimates in order to get sharp rates of nonlinear decay for solutions of (1.3), even for perturbations of the constant state. Further, pointwise estimates appear to be necessary in order to show *any* decay for general shocks and rarefactions [SX, LZ1, LZ2, SZ, L3].

As a further motivation, there is also an interesting connection between Eq. (1.1) and the Schrödinger equation, by which pointwise bounds on the Green's function of (1.1) lead to pointwise estimates on the time-propagator for appropriate Schrödinger potentials. And, while there have been a number of results published on direct evaluation and bounds in various norms on the energy dependent Green's function for the Laplace transformed Schrödinger equation, there are relatively few for the time propagator (see comments in [GS]). As usual, through the Feynman–Kac formula we can also obtain estimates on moment generating functions for the Brownian bridge process.

In the constant coefficient case of (1.1), or (1.2), a general method by way of Fourier analysis and Paley–Wiener estimates has been introduced for finding pointwise Green's function bounds [Ze, LZe, HZ1, HZ2], but no standard procedure for the nonconstant coefficient case has yet been put forward. Three notable approaches that have been brought to bear on this case are the refined parametrix method for shocks [SX] and rarefactions [SZ], the method of approximate Green's functions [L3], and the weighted norm approach of Sattinger [S, JGK]. However, the former two approaches are so far limited to the quasi-decoupled case of a constant identity viscosity coefficient and weak shock strength, while the latter applies essentially only in the scalar case (see discussion, [LZ2]).

The method introduced here employs the spectral approach of [LZe], extending it to the nonconstant coefficient case using the semigroup framework of [S, JGK] (but without weighted norms). This method of analysis works for nonconstant viscosity coefficient and arbitrary shock strength.

Our assumptions, made throughout the paper, will be as follows:

(I) The convection function, $a(x)$, and the viscosity function, $b(x)$, satisfy, for all $k \leqslant K$, $K \geqslant 1$:

(i) $a(x), b(x) \in C^k(\mathbb{R})$

(ii) $|(\partial^k/\partial x^k)(a(x)-a_\pm)|, |(\partial^k/\partial x^k)(b(x)-b_\pm)| = \mathbf{O}(e^{-\alpha|x|})$, $\alpha > 0$,

(iii) $a_\pm \neq 0$,

(iv) $b(x) \geqslant b_0 > 0$,

where $\lim_{x\to\pm\infty} a(x) = a_\pm$ and $\lim_{x\to\pm\infty} b(x) = b_\pm$.

(II) All eigenvalues, denoted by λ, of the operator L, defined by

$$Lv := (b(x)\, v_x)_x - (a(x)\, v)_x \qquad \text{in the case Eq. (1.1),} \tag{1.4}$$

or

$$Lv := b(x)\, v_{xx} - a(x)\, v_x \qquad \text{in the case of Eq. (1.2)} \tag{1.5}$$

lie in the strict negative-real half-plane, $\mathrm{Re}(\lambda) < 0$. Here, we may take as our space of eigenvalues any L^p space, $p < \infty$, so long as the eigenfunctions decay at $\pm\infty$.

(III) $W_y(0) \neq 0$, where $W_y(\lambda)$ is the Wronskian associated with L, as defined in Section 2.

Before stating our main result, we make a number of observations about these assumptions. Condition (I)(iv) corresponds to strict parabolicity, while (I)(iii), in the context of viscous shock waves, precludes degenerate "sonic" shocks, which also fail (I)(ii) since they decay as $1/x$ only [MN, N]. Conditions (II) and (III) restrict our attention to the case for which solutions v of (1.1)–(1.2) decay time-asymptotically to zero in L^2 norm. In condition (III), the Wronskian, $W_y(\lambda)$, is precisely the Evans function associated with the operator $L - \lambda$ [E1–E4, AGJ]. Though we do not prove it here, (III) is necessary and sufficient, given (II), for v to decay in time. This and other issues related to the Evans function are discussed at length in [ZH].

It should be noted that assumption (II) holds true in all cases of (1.1)–(1.2) except for Eq. (1.1) with $a_- > 0$ and $a_+ < 0$—the *Lax* case. This can be shown by a maximum principle argument in the case of Eq. (1.2) or in the case of Eq. (1.1) by the L^1 contraction principle. It is not difficult to see that either of these principles implies that all eigenvalues must have negative real parts, with the possible exception of the origin. Observing that the exact solution of the zero eigenvalue equation is either $\exp(\int_0^x (a/b)\, d\xi)$ in the unintegrated case or else the integral of this function in the integrated case, we see that this is bounded precisely in the unintegrated, *Lax* case of Eq. (1.1). A similar argument gives that condition (III) holds in all cases except for the unintegrated *Lax* case (because in this case zero is an eigenvalue) and its adjoint, the integrated expansive case ($a_- < 0$, $a_+ > 0$). These omitted cases exhibit only *bounded stability* of solutions, and consequently have more complicated Green's functions. Such cases can be treated by similar methods, but at the expense of further effort [ZH].

A consequence of Assumptions (I) and (II) (see Lemma 3.3) is that the entire point spectrum of L must lie strictly to the left of a parabola in the complex plane opening to the left and crossing the real axis at a negative number, say, $-d$. We will call this contour Γ_c and write it as

$$\lambda_c(k) = -ck^2 - d + ik, \tag{1.6}$$

where $c, d \in \mathbb{R}^+$, $c < \min(|a_-|, |a_+|)$, $d < \min(a_-^2/4b_-, a_+^2/4b_+)$. The goal of this paper is, with these three assumptions made, to prove the following theorem:

THEOREM 1.1. *Under assumptions* (I), (II), *and* (III) *and for some constants, C, C_n, M, $n \leqslant K$ and $\delta > 0$ depending on the asymptotic behavior of $a(x)$ and $b(x)$ and also on the eigenvalues of L, that is, the values of c and d in* (1.6), *the Green's function $G(t, x; y)$ for Eq.* (1.1), *or* (1.2), *satisfies the following estimates for* $(x > 0)$ (*symmetric estimates hold in the case* $x < 0$):

(i) *For $y > 0$, $a_+ > 0$ and also for $x > y > 0$, $a_+ < 0$*

$$|G(t, x; y)| \leqslant \frac{Ce^{-(x-y-a_+t)^2/4tb_+M}}{\sqrt{tb_+}},$$

$$\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right| \leqslant \frac{C_n e^{-(x-y-a_+t)^2/4tb_+M}}{(tb_+)^{(n+1)/2}}.$$

(ii) *For $y > x > 0$, $a_+ < 0$*

$$|G(t, x; y)| \leqslant \frac{Ce^{-(x-y-a_+t)^2/4tb_+M}}{\sqrt{tb_+}},$$

$$\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right| \leqslant \frac{C_n e^{-(x-y-a_+t)^2/4tb_+M}}{\sqrt{tb_+}} e^{-\delta|x|} + \frac{C_n e^{-(x-y-a_+t)^2/4tb_+M}}{(tb_+)^{(n+1)/2}}.$$

The remaining cases are for $y < 0$.

(iii) *For $a_+ > 0$, $a_- < 0$*

$$|G(t, x; y)| \leqslant \frac{Ce^{-(x-y-a_+t)^2/4tb_+M}}{\sqrt{tb_+}} e^{-\delta|y|},$$

$$\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right| \leqslant \frac{C_n e^{-(x-y-a_+t)^2/4tb_+M}}{(tb_+)^{(n+1)/2}} e^{-\delta|y|}.$$

(iv) *For $a_+ < 0$, $a_- > 0$*

$$|G(t, x; y)| \leqslant \frac{Ce^{-(x-y-a_-t)^2/4tb_-M}}{\sqrt{tb_-}} e^{-\delta|x|},$$

$$\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right| \leqslant \frac{Ce^{-(x-y-a_-t)^2/4tb_-M}}{\sqrt{tb_-}} e^{-\delta|x|} + \frac{Ce^{-(x-y-a_-t)^2/4tb_-M}}{(tb_-)^{(n+1)/2}} e^{-\delta|x|}.$$

(v) *For* $a_+ > 0$, $a_- > 0$

$$|G(t, x; y)| \leqslant \frac{Ce^{-(x-(a_+/a_-)\,y-a_+t)^2/4tb_+M}}{\sqrt{tb_+}},$$

$$\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right| \leqslant \frac{C_n e^{-(x-(a_+/a_-)\,y-a_+t)^2/4tb_+M}}{(tb_+)^{(n+1)/2}} + \frac{C_n e^{-(x-(a_+/a_-)\,y-a_+t)^2/4tb_+M}}{\sqrt{tb_+}}.$$

(vi) *For* $a_+ < 0$, $a_- < 0$

$$|G(t, x; y)| \leqslant \frac{Ce^{-(x-y-a_-t)^2/4tb_-M}}{\sqrt{tb_-}},$$

$$\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right| \leqslant \frac{C_n e^{-(x-y-a_-t)^2/4tb_-M}}{(tb_-)^{(n+1)/2}}.$$

Before proceeding with the analysis we make a few remarks about this theorem and its applications. Note first that in each case the estimates consist simply of a Gaussian kernel centered about a path determined by the asymptotic values of the convection function $a(x)$. These estimates are sharp when compared to known exact solutions [Z, LZ1]. For short time, they reduce to the standard parabolic estimates of, e.g., [F], in which convection is neglected. The significance of the estimates of Theorem 1.1 is that they remain valid for *all* time, incorporating convective effects in the description of the central path. Such global estimates, especially the localization of the solutions, are essential in the study of nonlinear stability of viscous shock waves [LZ1–LZ2, SZ, L3].

It should be noted that the reduced algebraic decay in the derivative estimates of Cases (ii), (iv), and (v) is expected, as it agrees with exact known solutions. However, we also point out that in the noncompressive cases this reduced algebraic decay is contingent on the relative sizes of the asymptotic states of $a(x)$ and $b(x)$, an effect seen in the proof of Theorem 1.1 but not explicitly stated. For example, in Case (v) in which all mass is moving to the right, if $b_- = b_+$ and $a_+ < a_-$, mass will accumulate at the origin, leading to the diminished algebraic decay in time. On the other hand, if $a_+ > a_-$ then no mass accumlates and we see the additional algebraic decay in time. In general this effect seems to be governed by the relation (4.32).

This path-dependence on only the asymptotic values of $a(x)$ leads to some interesting observations. In the case with $a_- > 0$ and $a_+ > 0$, where the path is given by $x=(a_+/a_-)\,y+a_+t$, we see that the kernel moves, in general, with speed a_- while to the left of the origin and with speed a_+ while to the right of the origin. Thus its overall speed is a simple average as intuition would have us expect. Perhaps less intuitively, in Case (iii), where $a_+>0$, $a_-<0$ and $y<0$ we see the *possibility* for mass to move across the origin, though tempered by exponential y-decay—a phenomenon akin to quantum tunneling, hence a reminder of the connection between our equation and the Schrödinger equation. Finally, we remark that in the last case, $a_+<0$, $a_-<0$, since x, y and $-a_-t$ are all positive, the decay is (when the constant M is taken into account) path independent. We chose to state the path above in order to highlight the similarity between cases (iv) and (vi).

In Theorem 1.1 we have contented ourselves with stating derivative estimates only with respect to x. We justify this by noting that estimates on y-derivatives of $G(t, x; y)$ follow from the x-derivative estimates and the observation that if $G(t, x; y)$ is the Green's function for the equation $v_t=Lv$, then $G(t, y, x)$ is the Green's function for the equation $v_t=L^*v$, L^* the adjoint operator for L. Thus, since our approach works for both L and L^*, we need only estimate derivatives with respect to x.

Finally, we mention a few applications. First, we can recover the weighted norm results of [JGK] and the energy method results of [MN], extending each by obtaining *sharp* pointwise estimates on perturbations to scalar conservation laws with diffusion [H2]. Additionally, we have an alternate method of obtaining Liu's estimates in [L3]. Further, initial studies indicate that the same methods applied herein may be suitable for the analysis of equations with dispersion and higher order terms [HoZ]. Most importantly, the approach taken in the paper can be generalized to the case of systems, as shown in [ZH].

2. PRELIMINARIES

For definiteness we will carry out all computations in this paper for (1.1) only, as those for (1.2) follow similarly. Our approach will be to consider the eigenvalue equation

$$Lv=\lambda v, \tag{2.1}$$

where L has been defined in (1.4). In particular we solve the associated Green's function equation

$$(L-\lambda)\,v=-\delta_y(x). \tag{2.2}$$

If we let $R(\lambda) := (L-\lambda)^{-1}$ denote the *resolvent operator*, then (2.2) is solved by the Green's function

$$G_\lambda(x, y) = R(\lambda)\,\delta_y(x)$$

wherever $R(\lambda)$ is defined (whenever $\lambda \notin \sigma(L) :=$ spectrum of L).

The computation of $G_\lambda(x, y)$ is standard [CH] for the operator $L-\lambda$ in terms of the solutions of the eigenvalue ODE, (2.1). Our notation will be to let φ denote the (unique) decay modes associated with (2.1), so that φ^+ decays at $+\infty$ and φ^- decays at $-\infty$. On the other hand, ψ will denote the growth modes associated with (2.1), so that ψ^+ grows at $+\infty$ and ψ^- grows at $-\infty$. (Note that away from essential spectrum solutions always either grow or decay at $\pm\infty$ and that, for example, it may be the case that $\varphi^+ = \psi^-$.) Since growth modes are not unique we will choose a specific representative when necessary.

We can easily compute the asymptotic growth and decay rates of φ and ψ from (2.1) by noting that at $\pm\infty$ (2.1) becomes

$$b_\pm u_{xx} - a_\pm u_x - \lambda u = 0, \tag{2.3}$$

so that solutions of the form $u \sim e^{\mu x}$ give

$$b_\pm \mu^2 - a_\pm \mu - \lambda = 0.$$

This last equation can readily be solved for μ, leading to

$$\mu = \frac{a_\pm \pm \sqrt{a_\pm^2 + 4b_\pm \lambda}}{2b_\pm}.$$

We take the negative real axis as our branch cut for the radical, so that the real part of our radicals will always be positive. Our notation on μ will be $\mu_j^\pm$, where $\pm$ indicates which asymptotic value of $a(x)$ and $b(x)$ to use, and $\mathrm{Re}(\mu_1^\pm) < \mathrm{Re}(\mu_2^\pm)$, that is, $\mu_1^\pm$ represents the case in which the radical is subtracted and $\mu_2^\pm$ represents the case in which the radical is added. Throughout the analysis, we will make use of the observation that $\mu_j^+(\lambda)$ is analytic for all λ except on the negative real strip $\lambda < -a_+^2/(4b_+)$, and similarly that $\mu_j^-(\lambda)$ is analytic for all λ except on the negative real strip $\lambda < -a_-^2/(4b_-)$.

In terms of the above notation the Green's function $G_\lambda(x, y)$ for (2.1) becomes [CH]

$$G_\lambda(x, y) = \begin{cases} \dfrac{\varphi^+(x)\,\varphi^-(y)}{W(y)\,b(y)}, & x > y \\[2ex] \dfrac{\varphi^+(y)\,\varphi^-(x)}{W(y)\,b(y)}, & x < y, \end{cases} \tag{2.4}$$

where $W(y)$ denotes the usual Wronskian,

$$W(y)=\varphi^{+\prime}(y)\,\varphi^{-}(y)-\varphi^{+}(y)\,\varphi^{-\prime}(y) \tag{2.5}$$

and consequently satisfies Abel's equation,

$$W'(y)=\left(\frac{a(y)}{b(y)}-\frac{b'(y)}{b(y)}\right)W(y). \tag{2.6}$$

We note here that in the scalar case the Wronskian is precisely the Evans function.

Finally, we will achieve the desired estimate on $G(t, x; y)$ from Dunford's Integral (the resolvent formula for the semigroup) [Y], which gives

$$G(t, x; y)=\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t} G_{\lambda}(x, y)\, dx, \tag{2.7}$$

where Γ is a contour enclosing the entire spectrum of L (possibly passing through the point at ∞). The verification of Dunford's Integral is straightforward for our $G_{\lambda}(x, y)$ and the proof is here omitted. The interested reader is referred to Lemma 3.6 of [H1].

Before beginning the analysis we make a brief remark about notation. In all that follows, the terms $\mathbf{O}(\cdot)$ will be uniform in all variables other than the argument. Constants, C, will be independent of x, y, t and λ, but will often change without comment or relabeling from one expression to the next. We also note that the values of c and d for the contour Γ_c will be chosen during the course of the proof of Theorem 1.1, chosen smaller than the values given above so that at all times Γ_c will be an appropriate bound on the spectrum of the operator, L (see Lemma 3.3). Finally, our notation for the Wronskian will vary between $W_{\lambda}(y)$ and $W_{y}(\lambda)$, depending upon which variable is under discussion.

3. BOUNDS ON $\mathbf{G}_{\lambda}(\mathbf{x}, \mathbf{y})$

In this section we state five lemmas fundamental to the proof of Theorem 1.1, rendering proofs for the last three. The first two lemmas pertain to the behavior of the solutions, φ and ψ, of the ODE (2.1), and as their proofs entail no new ideas, they are not included. The third lemma addresses the behavior of the Wronskian (or Evans function), especially its analyticity and its set of zeros, and the final two give bounds on the Green's function, $G_{\lambda}(x, y)$ for (2.1).

In particular, Lemma 3.1 gives the existence of growth and decay mode solutions, ψ and φ, to (2.1) that are analytic in λ (for an appropriate region of λ). The lemma also precisely specifies the asymptotic behavior of these modes. The arguments needed in the proof of Lemma 3.1 are similar to those in [C, JGK], and the proof is rendered in its entirety in [H1].

LEMMA 3.1. *Let $|\lambda| < M_s$ for some constant M_s, and also let λ lie on or to the right of Γ_c. Under assumptions* (I), (II), *and* (III), *there exist solutions of* (2.1), *φ and ψ, satisfying the following asymptotic estimates ($n \leqslant K$; φ^+, ψ^+ for $x>0$; φ^-, ψ^- for $x<0$):*

$$\text{(i)} \qquad \varphi^+(x) = e^{\mu_1^+ x}(1 + \mathbf{O}(e^{-\alpha|x|})),$$

$$\frac{\partial^n}{\partial x^n}\varphi^+(x) = e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|})).$$

$$\text{(ii)} \qquad \varphi^-(x) = e^{\mu_2^- x}(1 + \mathbf{O}(e^{-\alpha|x|})),$$

$$\frac{\partial^n}{\partial x^n}\varphi^-(x) = e^{\mu_2^- x}((\mu_2^-)^n + \mathbf{O}(e^{-\alpha|x|})).$$

$$\text{(iii)} \qquad \psi^+(x) = e^{\mu_2^+ x}(1 + \mathbf{O}(e^{-\alpha|x|})),$$

$$\frac{\partial^n}{\partial x^n}\psi^+(x) = e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|})).$$

$$\text{(iv)} \qquad \psi^-(x) = e^{\mu_1^- x}(1 + \mathbf{O}(e^{-\alpha|x|})),$$

$$\frac{\partial^n}{\partial x^n}\psi^-(x) = e^{\mu_1^- x}((\mu_1^-)^n + \mathbf{O}(e^{-\alpha|x|})).$$

Moreover, $\varphi^\pm$ and $\psi^\pm$ are analytic in λ for λ on or to the right of Γ_c. ∎

We now state a lemma that gives large $|\lambda|$ estimates on φ and ψ. The proof hinges on a rescaling argument similar to those of [AGJ, GZ] and appears in [H1].

LEMMA 3.2. *Under assumptions* (I), (II), *and* (III), *φ^+ and φ^- satisfy the following estimates in λ: For λ in the intersection of a large enough ball around the origin, say $|\lambda| > M_l$, and the set on or to the right of Γ_c, we have*

$$\varphi^\pm(x) = k^\pm(x)(1 + \mathbf{O}(|\lambda|^{-1/2})), \qquad x \in \mathbb{R}, \quad k^\pm \in C(\mathbb{R}), \quad \textit{bounded in } \lambda,$$

$$|k^\pm(x)| \neq 0$$

$$\frac{\partial^k}{\partial x^k}\varphi^\pm(x) = (\mp\sqrt{\lambda/b(x)})^k\, k^\pm(x)(1 + \mathbf{O}(|\lambda|^{-1/2})), \qquad x \in \mathbb{R}.$$

We next observe as in [JGK], that the Wronskian (or Evans function), $W_y(\lambda)$, is analytic for the domain of λ we are considering and hence therein has a discrete set of zeros. Bounds on the point spectrum follow directly from this.

LEMMA 3.3. *Under assumptions* (I), (II), *and* (III), *we have* (1) *for λ on or to the right of Γ_c, $W_y(\lambda)$ is analytic in λ, and* (2) *for c and d appropriately chosen in* (1.6), *the zeroes of $W_y(\lambda)$ lie strictly to the left of Γ_c.*

Proof. Part (1) is immediate since analyticity of $W_y(\lambda)$ comes directly through (2.5) from the analyticity in λ of $\varphi^{\pm}(y)$ and $\varphi^{\pm\prime}(y)$.

As for (2) the essential spectrum of L as defined in either (1.4) or (1.5) is always bounded on the right by a parabola opening to the left and passing through the origin, specifically the widest of the family of four parabolas, $\mathrm{Re}(\mu_j^{\pm})=0$, $j=1, 2$. Thus any zeros of the Wronskian, $W_\lambda(y)$, lying to the right of this parabola must be point spectrum and consequently eigenvalues of L, limiting them to the negative real half-plane. Further, there can be only finitely many of these zeros in a ball around the origin, because, by (1), in the domain of eigenvalues under consideration, the Wronskian is a non-zero analytic function of λ and hence can have only isolated zeros in any bounded neighborhood. An energy estimate, or the large $|\lambda|$ estimate of Lemma 3.5, suffices to show that all such zeros are confined to a bounded domain. Also, by assumption (III), $W_y(0)\neq 0$ so there is a neighborhood around $\lambda=0$ in which $W_\lambda(y)\neq 0$. Therefore, we can enclose all zeros of $W_y(\lambda)$ by a parabola in the negative half-plane that does not pass through the origin. We choose c and d so that Γ_c lies to the right of this parabola. ∎

The final two lemmas of this section pertain to the elliptic Green's function, $G_\lambda(x, y)$.

LEMMA 3.4 (Small $|\lambda|$ estimates for the elliptic Green's function). *Under assumptions* (I), (II), *and* (III) *and for $|\lambda|$ bounded above, say $|\lambda|\leqslant M_s$ for some constant M_s, and λ on or to the right of Γ_c we get the following estimates on the Green's function for* (2.1):

(i) $x>y>0$,

$$G_\lambda(x, y)=\frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{\mu_1^+(x-y)}$$

$$\frac{\partial^n}{\partial x^n}G_\lambda(x, y)=\frac{\mathbf{O}(1)}{W_0(\lambda)}\,(\mu_1^+)^n\, e^{\mu_1^+(x-y)}+\frac{\mathbf{O}(e^{-\alpha|x|})}{W_0(\lambda)}\, e^{\mu_1^+(x-y)}$$

(ii) $y > x > 0$,

$$G_\lambda(x, y) = \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{\mu_2^+(x-y)}$$

$$\frac{\partial^n}{\partial x^n} G_\lambda(x, y) = \frac{\mathbf{O}(1)}{W_0(\lambda)}\, (\mu_2^+)^n\, e^{\mu_2^+(x-y)} + \frac{\mathbf{O}(e^{-\alpha|x|})}{W_0(\lambda)}\, e^{\mu_2^+(x-y)}$$

(iii) $x > 0 > y$,

$$G_\lambda(x, y) = \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{\mu_1^+ x} e^{-\mu_1^- y}$$

$$\frac{\partial^n}{\partial x^n} G_\lambda(x, y) = \frac{\mathbf{O}(1)}{W_0(\lambda)}\, (\mu_1^+)^n\, e^{\mu_1^+ x} e^{-\mu_1^- y} + \frac{\mathbf{O}(e^{-\alpha|x|})}{W_0(\lambda)}\, e^{\mu_1^+ x} e^{-\mu_1^- y},$$

with symmetric estimates for $x < 0$. Notice that for $|\lambda|$ bounded, $W_0(\lambda)$ must be bounded away from 0 *since we have no point spectrum to the right of Γ_c.*

Proof. Since we have assumed $|\lambda|$ bounded we are at liberty to apply Lemma 3.1. Consider first Case (i) for which from (2.4),

$$G_\lambda(x, y) = \frac{\varphi^+(x)\, \varphi^-(y)}{W(y)\, b(y)}.$$

Since Lemma 3.1 does not give us an estimate on $\varphi^-(y)$ for $y > 0$ we must write $\varphi^-(y)$ as a linear combination of $\varphi^+(y)$ and $\psi^+(y)$. That is,

$$\varphi^-(y) = \psi^+(y) + B(\lambda)\, \varphi^+(y), \qquad y > 0,$$

where $B(\lambda)$ is $\mathbf{O}(1)$ in λ as long as $|\lambda|$ is bounded, and we have scaled out the coefficient in front of $\psi^+(y)$ by appropriately scaling $\varphi^-(y)$.

According then to Lemma 3.1

$$\begin{aligned}\varphi^-(y) &= e^{\mu_2^+ y}(1 + \mathbf{O}(e^{-\alpha_2^+|y|})) + B(\lambda)\, e^{\mu_1^+ y}(1 + \mathbf{O}(e^{-\alpha_1^+|y|}))\\ &= e^{\mu_2^+ y}[1 + \mathbf{O}(e^{-\alpha_2^+|y|}) + B(\lambda)\, e^{(\mu_1^+ - \mu_2^+) y}(1 + \mathbf{O}(e^{-\alpha_1^+|y|}))]\\ &= e^{\mu_2^+ y}\mathbf{O}(1).\end{aligned}$$

By Lemma 3.1, $\varphi^+(x) = e^{\mu_1^+ x}\mathbf{O}(1)$ so that (with $1/b(y) = \mathbf{O}(1)$)

$$G_\lambda(x, y) = \frac{e^{\mu_1^+ x} e^{\mu_2^+ y}\mathbf{O}(1)}{W(y)}.$$

Recalling that $W(y) = W_0(\lambda)\, e^{\int_0^y (a(s)/b(s) - b'(s)/b(s))\, ds}$, we get

$$G_\lambda(x, y) = \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{-\int_0^y (a(s)/b(s) - b'(s)/b(s))\, ds} e^{\mu_1^+(x-y)} e^{(\mu_1^+ + \mu_2^+)\, y}.$$

We note that $\mu_1^+ + \mu_2^+ = (a_+ - \sqrt{a_+^2 + 4b_+\lambda})/2b_+ + (a_+ + \sqrt{a_+^2 + 4b_+\lambda})/2b_+ = a_+/b_+$, so that we get

$$G_\lambda(x, y) = \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{-\int_0^y ((a(s)/b(s) - b'(s)/b(s)) - a_+/b_+)\, ds} e^{\mu_1^+(x-y)}.$$

By assumption (I), $e^{-\int_0^y ((a(s)/b(s) - b'(s)/b(s)) - a_+/b_+)\, ds} = \mathbf{O}(1)$, leading to the claimed estimate.

Next we consider Case (ii), returning shortly to the derivatives. Now, $G_\lambda(x, y) = \varphi^+(y)\, \varphi^-(x)/W(y)\, b(y)$ so that we must use

$$\varphi^-(x) = e^{\mu_2^+ x}\mathbf{O}(1).$$

Hence we get

$$G_\lambda(x, y) = \frac{e^{\mu_1^+ y} e^{\mu_2^+ x}\mathbf{O}(1)}{W(y)} = \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{\mu_2^+(x-y)},$$

in the same manner as above.

In Case (iii) we get

$$\begin{aligned}
G_\lambda(x, y) &= \frac{e^{\mu_1^+ x} e^{\mu_2^- y}}{W(y)}\, \mathbf{O}(1) \\
&= \frac{\mathbf{O}(1)\, e^{\mu_1^+ x} e^{((a_- + \sqrt{a_-^2 + 4b_-\lambda})/2b_-)\, y}}{W_0(\lambda)}\, e^{-\int_0^y (a(s)/b(s) - b'(s)/b(s))\, ds} \\
&= \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{\mu_1^+ x} e^{((-a_- + \sqrt{a_-^2 + 4b_-\lambda})/2b_-)\, y + (a_-/b_-)\, y} \\
&\quad \times e^{-\int_0^y (a(s)/b(s) - b'(s)/b(s))\, ds} \\
&= \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{\mu_1^+ x} e^{-\mu_1^- y} e^{-\int_0^y (a(s)/b(s) - b'(s)/b(s)) - (a_-/b_-)\, ds} \\
&= \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{\mu_1^+ x} e^{-\mu_1^- y}.
\end{aligned}$$

We now prove the three derivative estimates associated with the cases analyzed above. For Case (i) we have again $G_\lambda(x, y) = \varphi^+(x)\,\varphi^-(y)/W(y)$ so that

$$\frac{\partial^n}{\partial x^n} G_\lambda(x, y) = \frac{[(\partial^n/\partial x^n)\,\varphi^+(x)]\,\varphi^-(y)}{W(y)\,b(y)},$$

which by virtue of Lemma 3.1 is

$$\begin{aligned}\frac{\partial^n}{\partial x^n} G_\lambda(x, y) &= \frac{e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))\,e^{\mu_2^+ y}\mathbf{O}(1)}{W(y)}\\ &= \frac{\mathbf{O}(1)\,e^{\mu_1^+ x}e^{\mu_2^+ y}(\mu_1^+)^n}{W(\lambda)} + \frac{\mathbf{O}(e^{-\alpha|x|})\,e^{\mu_1^+ x}e^{\mu_2^+ y}}{W(\lambda)}.\end{aligned}$$

The Wronskian is dealt with as before, leading to the claimed estimate.

Case (ii) involves one new aspect so we include details even though it is very similar to Case (i). The problem again is that for $y > x > 0$ we have

$$G_\lambda(x, y) = \frac{\varphi^+(y)\,\varphi^-(x)}{W(y)\,b(y)}$$

and no information from Lemma 3.1 about the behavior of $\varphi^-(x)$ for $x > 0$.

As before we resolve this by writing $\varphi^-(x)$ in terms of the growing and decaying modes at $+\infty$. That is, we write

$$\varphi^-(x) = \psi^+(x) + B(\lambda)\,\varphi^+(x), \qquad x > 0,$$

where $B(\lambda) = \mathbf{O}(1)$ in λ (for $|\lambda|$ bounded). Thus we get

$$\frac{\partial^n}{\partial x^n}\varphi^-(x) = \frac{\partial^n}{\partial x^n}\psi^+(x) + B(\lambda)\,\frac{\partial^n}{\partial x^n}\varphi^+(x),$$

which, according to Lemma 3.1, becomes

$$\begin{aligned}\frac{\partial^n}{\partial x^n}\varphi^-(x) &= e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))\\ &\qquad + B(\lambda)\,e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))\\ &= e^{\mu_2^+ x}[(\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))\\ &\qquad + B(\lambda)\,e^{(\mu_1^+ - \mu_2^+)x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|})]\\ &= (\mu_2^+)^n\,e^{\mu_2^+ x}\mathbf{O}(1) + e^{\mu_2^+ x}\mathbf{O}(e^{-\alpha|x|}).\end{aligned}$$

Now,

$$\frac{\partial^n G_\lambda(x, y)}{\partial x^n} = \frac{\varphi^+(y)(\partial^n/\partial x^n)\,\varphi^-(x)}{W(y)\,b(y)}$$

so that

$$\frac{\partial^n G_\lambda(x, y)}{\partial x^n} = \frac{\mathbf{O}(1)(\mu_2^+)^n\, e^{\mu_1^+ y} e^{\mu_2^+ x}}{W(y)} + \frac{\mathbf{O}(e^{-\alpha|x|})\, e^{\mu_1^+ y} e^{\mu_2^+ x}}{W(y)}.$$

Bringing the Wronskian into play in the usual manner leads to the claimed estimate.

Last, we analyze the case $x > 0 > y$ for which, proceeding as above, we get

$$\frac{\partial^n}{\partial x^n} G_\lambda(x, y) = \frac{(\partial^n/\partial x^n)\,\varphi^+(x)\,\varphi^-(y)}{W(y)\,b(y)},$$

which by Lemma 3.1 becomes

$$\frac{\partial^n}{\partial x^n} G_\lambda(x, y) = \frac{e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))\, e^{\mu_2^- y}\mathbf{O}(1)}{W(y)}.$$

Taking the Wronskian into account as above we arrive at the derivative estimate of (iii). ▮

LEMMA 3.5 (Large $|\lambda|$ estimates for the elliptic Green's function). *Under assumptions* (I), (II), *and* (III) *and for* $|\lambda|$ *in the intersection of a large enough ball around the origin, say* $|\lambda| \geqslant M_1$, *and the set on or to the right of* Γ_c, *with also* $n \leqslant K$, *we have the following estimates on* $G_\lambda(x, y)$ ($b_s := \sup_{s \in [x, y]} b(s)$):

$$\left|\frac{\partial^n}{\partial x^n} G_\lambda(x, y)\right| \leqslant \mathbf{O}(|\lambda|^{(n-1)/2})\, e^{-\mathrm{Re}(\sqrt{\lambda/b_s}/2)\,|x-y|}.$$

We remark before proving this theorem that here we get the same result for all three $x > 0$ cases of Lemma 3.4 as well as for $x < 0$.

Proof. Because of the similarity in the arguments we will only give here an analysis of the single case $y > x > 0$. From (2.4) for the case $x < y$ we have

$$G_\lambda(x, y) = \frac{\varphi^-(x)\,\varphi^+(y)}{W(y)\,b(y)},$$

which can be rewritten as

$$G_\lambda(x, y) = \frac{\varphi^-(x)}{\varphi^-(y)} \cdot \frac{\varphi^-(y)\,\varphi^+(y)}{W(y)\,b(y)}.$$

We first show that $\varphi^-(y)\,\varphi^+(y)/W(y)\,b(y)$ is bounded above. For large $|\lambda|$ we have, according to (2.5) and Lemma 3.2,

$$\begin{aligned} W(y) &= -\sqrt{\lambda/b(y)}\,k^+(y)(1+\mathbf{O}(|\lambda|^{-1/2})\,k^-(y)(1+\mathbf{O}(|\lambda|^{-1/2})) \\ &\quad - k^+(y)(1+\mathbf{O}(|\lambda|^{-1/2})(\sqrt{\lambda/b(y)})\,k^-(y)(1+\mathbf{O}(|\lambda|^{-1/2})) \\ &= -2\sqrt{\lambda/b(y)}\,k^+(y)\,k^-(y)(1+\mathbf{O}(|\lambda|^{-1/2})). \end{aligned}$$

Therefore we have

$$\begin{aligned} \left|\frac{\varphi^-(y)\,\varphi^+(y)}{W(y)}\right| &= \left|\frac{k^-(y)\,k^+(y)(1+\mathbf{O}(|\lambda|^{-1/2}))}{2b(y)\sqrt{\lambda/b(y)}\,k^-(y)\,k^+(y)(1+\mathbf{O}(|\lambda|^{-1/2}))}\right| \\ &= \mathbf{O}(|\lambda|^{-1/2}). \end{aligned}$$

We next bound the growth of $\varphi^-(x)/\varphi^-(y)$. According to Lemma 3.2 $\varphi^{-\prime}(x)$ can only differ from $\varphi^-(x)$ by a term of the form $\sqrt{\lambda/b(x)}$ $(1+\mathbf{O}(|\lambda|^{-1/2}))$. Thus we have the relation

$$\varphi^{-\prime}(x) = \sqrt{\lambda/b(x)}\,\varphi^-(x)(1+\mathbf{O}(|\lambda|^{-1/2})),$$

an ordinary differential equation. We can solve this ODE in terms of initial data given at $x = y$ to get

$$\varphi^-(x) = \varphi^-(y)\,e^{\int_y^x \sqrt{\lambda/b(s)}\,(1+\mathbf{O}(|\lambda|^{-1/2}))\,ds},$$

where $\mathbf{O}(|\lambda|^{-1/2})$ has (bounded) s dependence. We get then for $x < y$

$$\left|\frac{\varphi^-(x)}{\varphi^-(y)}\right| = e^{\int_y^x \mathrm{Re}\,\sqrt{\lambda/b(s)}\,(1+\mathbf{O}(|\lambda|^{-1/2}))\,ds} \leqslant e^{-\mathrm{Re}\,\sqrt{\lambda/b_s}\,|x-y|\,(1-\mathbf{O}(|\lambda|^{-1/2}))}.$$

Thus, for $|\lambda|$ large enough, we have

$$\left|\frac{\varphi^-(x)}{\varphi^-(y)}\right| \leqslant e^{-\mathrm{Re}\,\sqrt{\lambda/b_s}/2\,|x-y|}.$$

We conclude that in this large $|\lambda|$ region we have

$$|G_\lambda(x, y)| \leqslant \mathbf{O}(|\lambda|^{-1/2})\,e^{-\mathrm{Re}\,\sqrt{\lambda/b_s}/2\,|x-y|},$$

where the 2 could be any constant larger than 1, depending on how large $|\lambda|$ is to be taken initially, that is, on how large M_l is to be taken.

The x-derivatives follow immediately since

$$\frac{\partial^k}{\partial x^k} G_\lambda(x, y) = \frac{(\partial^k/\partial x^k)\, \varphi^-(x)\, \varphi^+(y)}{W(y)}$$
$$= \frac{(\partial^k/\partial x^k)\, \varphi^-(x)}{(\partial^k/\partial y^k)\, \varphi^-(y)} \cdot \frac{((\partial^k/\partial y^k)\, \varphi^-(y))\, \varphi^+(y)}{W(y)},$$

and the $(\partial^k/\partial x^k)\, \varphi^-(x)$ are given in Lemma 3.2 as

$$\frac{\partial^k}{\partial x^k} \varphi^-(x) = (\sqrt{\lambda/b(x)})^k\, k^-(x)(1 + \mathbf{O}(|\lambda|^{-1/2})),$$

so that we have

$$\frac{d}{dx} \frac{\partial^k}{\partial x^k} \varphi^-(x) = \sqrt{\lambda/b(x)}\, \frac{\partial^k}{\partial x^k} \varphi^-(1 + \mathbf{O}(|\lambda|^{-1/2})).$$

This is an ODE for $\varphi^{-^{(k)}}(x)$ with solution

$$\frac{\partial^k}{\partial x^k} \varphi^-(x) = \frac{\partial^k}{\partial y^k} \varphi^-(y)\, e^{\int_y^x \sqrt{\lambda/b(s)}\, (1 + \mathbf{O}(|\lambda|^{-1/2}))\, ds},$$

which as above yields the estimate

$$\left| \frac{(\partial^k/\partial x^k)\, \varphi^-(x)}{(\partial^k/\partial y^k)\, \varphi^-(y)} \right| \leqslant e^{-\mathbf{Re}(\sqrt{\lambda/b_s}/2)\, |x-y|}.$$

Also,

$$\left| \frac{((\partial^k/\partial y^k)\, \varphi^-(y))\, \varphi^+(y)}{W(y)} \right| = \left| \frac{(\sqrt{\lambda/b(y)})^k\, k_\varphi^-(y)\, k_\varphi^+(y)(1 + \mathbf{O}(|\lambda|^{-1/2}))}{2b(y)\, \sqrt{\lambda/b(y)}\, k_\varphi^-(y)\, k_\varphi^+(y)(1 + \mathbf{O}(|\lambda|^{-1/2}))} \right|$$
$$= \mathbf{O}(|\lambda|^{(k-1)/2})$$

so that

$$\left| \frac{\partial^k}{\partial x^k} G_\lambda(x, y) \right| \leqslant \mathbf{O}(|\lambda|^{(k-1)/2})\, e^{-\mathbf{Re}(\sqrt{\lambda/b_s}/2)\, |x-y|},$$

finishing off the case, $y > x > 0$. ∎

4. PROOF OF MAIN THEOREM

Proof of Theorem 1.1. *Case* (i) *overview.* As the analysis of each subcase of (i) is similar, we give the details only in the single subcase, $x > y > 0$, $a_+ > 0$, $a_- < 0$. According to Lemma 3.4 the Green's function for $L - \lambda I$ in this case is

$$G_\lambda(x, y) = \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_1^+(x-y)}$$

for $|\lambda|$ bounded by M_s. Therefore by Dunford's integral we have

$$G(t, x; y) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_1^+(x-y)} \, d\lambda, \tag{4.1}$$

as long as there exists a contour Γ surrounding the spectrum of L that remains inside the M_s-ball.

We must do two things to get an estimate on $G(t, x; y)$. First, we need to choose an appropriate contour along which Dunford's integral can either be evaluated explicitly or sharply estimated, and, second, we must avoid the region of eigenvalues that are bounded to the left of Γ_c.

So as to limit our discussion to choosing an appropriate contour, we will first ignore the point spectrum and the fact that our estimates change for large $|\lambda|$. Motivated by an analysis of the constant coefficient case, we use the same contour appropriate there, denoted by Γ_+. Parametrized by the real variable, k, Γ_+ has the form

$$\lambda_+(k) = -b_+(k + i\alpha_+)^2 - ia_+(k + i\alpha_+), \tag{4.2}$$

where $\alpha_+ := (x - y - a_+ t)/2b_+ t$. Along this contour, it can easily be verified that μ_1^+ satisfies

$$\mu_1^+|_{\Gamma_+} = -\alpha_+ + ik. \tag{4.3}$$

A brief analysis along Γ_+ will allow us to investigate the basic approach taken in the proof of Theorem 1.1 in a simple setting. Taking the absolute value of (4.1) along Γ_+ yields

$$|G(t, x; y)| \leqslant \frac{1}{2\pi} \int_{\Gamma_+} \left| \frac{\mathbf{O}(1)}{W_0(\lambda)} \right| e^{\mathrm{Re}(\lambda t + \mu_1^+(x-y))} \, |d\lambda|,$$

where, on Γ_+,

$$\mathrm{Re}(\lambda t + \mu_1^+(x - y)) = -b_+ k^2 t - \alpha_+^2 t b_+.$$

Hence, we have

$$|G(t, x; y)| \leqslant \frac{e^{-\alpha_+^2 t b_+}}{2\pi} \int_{-\infty}^{+\infty} |\mathbf{O}(1)|\, e^{-b_+ k^2 t} \frac{|-2b_+(k + i\alpha_+) - ia_+|}{|W_0(\lambda_+(k))|}\, dk.$$

We note that the term $|-2b_+(k+i\alpha_+)-ia_+|/|W_0(\lambda_+(k))|$ is $\mathbf{O}(1)$ in k since, for y fixed, we have seen in the proof of Lemma 3.5 that $W_0(\lambda) = \mathbf{O}(\sqrt{|\lambda|}) = \mathbf{O}(k)$ for large values of $|\lambda|$. Thus for $|\alpha_+|$ bounded above (usually associated with $|\lambda|$ bounded—the estimate employed here) we have

$$|G(t, x; y)| \leqslant C \frac{e^{-\alpha_+^2 t b_+}}{\sqrt{t b_+}} = C \frac{e^{-(x-y-a_+ t)^2/4b_+ t}}{\sqrt{t b_+}}, \tag{4.4}$$

as expected.

The difference between (4.4) and the claim from Theorem 1.1 is that in (4.4) the exponential decay rate is not reduced by the constant M. In the following analysis we will see explicitly how the value of M depends on where the point spectrum lies, that is, on the constants c and d.

We begin by modifying Γ_+ in such a way that it avoids the allowed point spectrum. The forthcoming exposition will be clarified by Figs. 4.1 and 4.2.

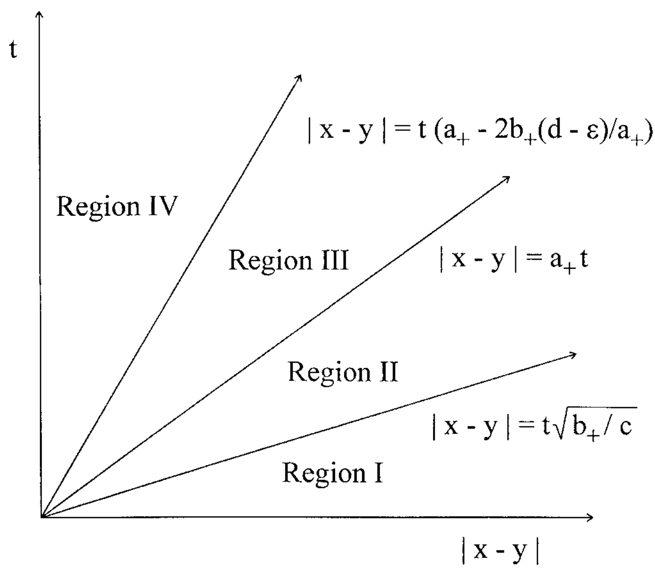

FIG. 4.1. Regions I–IV.

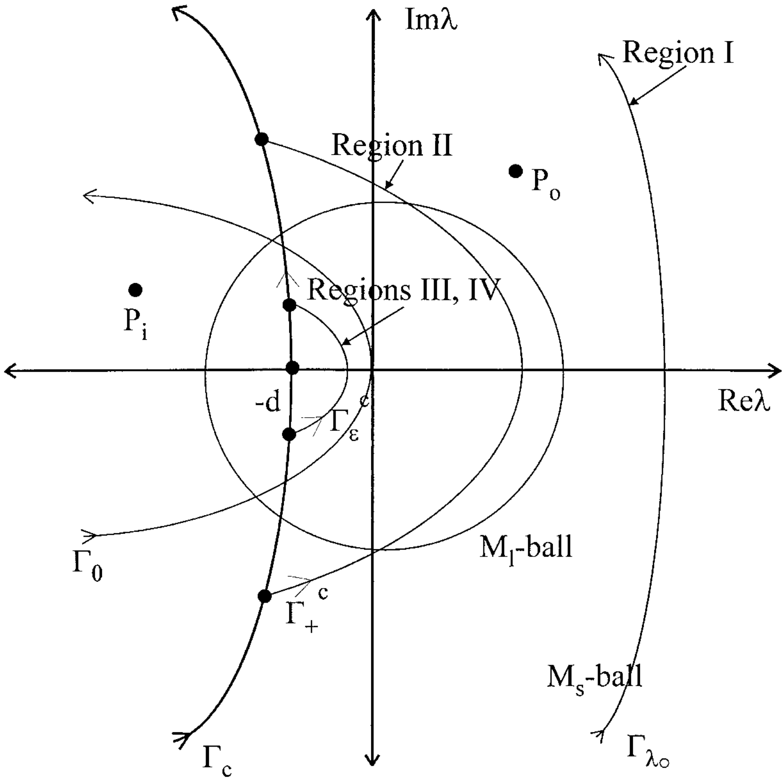

FIG. 4.2. Contours in the complex plane.

We note that along Γ_+ we have (from (4.2))

$$\operatorname{Re} \lambda_+ = -b_+ \left(\frac{t}{x-y}\right)^2 (\operatorname{Im} \lambda_+)^2 + \frac{(x-y)^2 - (a_+ t)^2}{4t^2 b_+}. \tag{4.5}$$

Case (i), *Region* I. We will look at each of the four regions in Fig. 4.1, beginning with Region I in which we have the inequality

$$|x-y| > t\sqrt{b_+/c}. \tag{4.6}$$

The Region I analysis will be similar for each case of Theorem 1.1, so we will work through it once carefully, and later often refer back. We see from (4.5) and (4.6) that c may be chosen small enough so that each contour in Region I lies entirely outside the M_l-ball, crossing the real axis to the right of this ball. (In Fig. 4.2, Γ_{λ_0} is an example of such a Region I contour.)

Thus our analysis of Region I will begin by combining Lemma 3.5 and Dunford's integral to get

$$|G(t, x; y)| \leqslant C \int_{\Gamma_{\lambda_0}} e^{\mathrm{Re}(\lambda t)} \mathbf{O}(|\lambda|^{-1/2})\, e^{-(\mathrm{Re}\sqrt{\lambda/b_s})/2\,|x-y|}\,|d\lambda|, \tag{4.7}$$

where Γ_{λ_0} is a contour to be appropriately chosen below. Inequality (4.6) leads us to expect that a new contour can be found (also lying entirely outside the M_l-ball) in which the $|x-y|$ exponent will dominate the integrand of (4.7). Since Region I is the small-t region, we expect the diffusion term to dominate so that we get behavior akin to that of the heat equation. Motivated by this observation, we take in Region I the contour that arises when analyzing the equation, $u_t = (b(x)\, u_x)_x$ by the method of Laplace transforms with respect to t. This contour, denoted by Γ_{λ_0}, has the form ($\mathrm{Re}\sqrt{\lambda/b_s} = \sqrt{\lambda_0}$)

$$\lambda(k) = b_s(\lambda_0 - k^2) + b_s 2ik\sqrt{\lambda_0}, \tag{4.8}$$

from which we obtain

$$\mathrm{Re}\,\lambda = -\left(\frac{1}{4\lambda_0 b_s}\right)(\mathrm{Im}\,\lambda)^2 + \lambda_0 b_s. \tag{4.9}$$

In (4.9) λ_0 must be chosen in such a way that Γ_{λ_0} will remain outside the M_l-ball where Lemma 3.5 is valid.

Along the contour Γ_{λ_0} we have

$$\begin{aligned}
&\int_{\Gamma_{\lambda_0}} \mathbf{O}(|\lambda|^{-1/2})\, e^{\mathrm{Re}(\lambda t)} e^{-\mathrm{Re}\,(\sqrt{\lambda/b_s})/2\,|x-y|}\,|d\lambda| \\
&\quad\leqslant \int_{-\infty}^{+\infty} \mathbf{O}(|\lambda(k)|^{-1/2})\, e^{b_s\lambda_0 t - b_s k^2 t} e^{-\sqrt{\lambda_0}/2\,|x-y|}\,|-2b_s k + 2ib_s\sqrt{\lambda_0}|\,dk \\
&\quad\leqslant C e^{b_s\lambda_0 t - \sqrt{\lambda_0}/2\,|x-y|} \int_{-\infty}^{+\infty} \mathbf{O}(|\lambda(k)|^{-1/2})\,|-2b_s k + 2ib_s\sqrt{\lambda_0}|\, e^{-b_s k^2 t}\,dk.
\end{aligned}$$

In order to obtain the claimed decay, we take $b_s\lambda_0 t = (\sqrt{\lambda_0}/4)\,|x-y|$ so that $\lambda_0 = |x-y|^2/16t^2 b_s^2$ is chosen. We need to ensure that the contour Γ_{λ_0} with this definition of λ_0 lies entirely outside the M_l-ball. But we have $|\lambda| = \sqrt{b_s^2(\lambda_0 - k^2)^2 + b_s^2 4k^2\lambda_0} = b_s(\lambda_0 + k^2)$, a quantity always larger than $b_s\lambda_0$. In Region I, $|x-y|/t\sqrt{b_+} > 1/\sqrt{c}$, so to get $b_s\lambda_0 = |x-y|^2/16t^2 b_s > M_l$, we need only take c small enough so that $16c(b_s/b_+)\,M_l < 1$. Also, from the above explicit expression for $|\lambda|$, $|\lambda|^{-1/2} \leqslant C/(\sqrt{\lambda_0}+|k|)$. Thus, continuing from (4.10), we get

$$Ce^{b_s t(|x-y|^2/16t^2b_s^2)-(|x-y|/8tb_s)\,|x-y|} \int_{-\infty}^{+\infty} \frac{|k|+\sqrt{\lambda_0}}{\sqrt{\lambda_0}+|k|}\, e^{-b_s k^2 t}\, dk$$

$$\leqslant C\frac{e^{-|x-y|^2/16tb_s}}{\sqrt{tb_s}}. \tag{4.11}$$

We then arrive at the exponential decay given in the statement of Case (i) through the computation

$$\begin{aligned} |x-y-a_+t| &\leqslant |x-y|+|a_+t| \\ &\leqslant |x-y|+a_+\sqrt{c/b_s}\,|x-y| \\ &\leqslant (1+a_+\sqrt{c/b_s})\,|x-y|. \end{aligned} \tag{4.11}$$

By changing the scaling constant, M, we can write the result in terms of b_+ rather than b_s so that it will match our analysis below in the region where Lemma 3.4 applies.

Case (i), *Region* II. In order to make the transition from Region I to Region II smooth we will now replace c in the definition of Γ_c with $c/2$. Since all Γ_+ contours not in Region I have quadratic coefficient less than $-c$, all contours in Region II will eventually intersect Γ_c. In replacing c with $c/2$ we have set a bound on how far from the origin the points of intersection can lie. Thus, since we may take M_s as large as we like, we will be able to take $M_s > M_l$ sufficiently, to ensure that in Region II the small $|\lambda|$ estimates are applicable until Γ_+ intersects Γ_c. In this way the analysis will be simplified, as it will only be on Γ_c that we will have to have both Lemma 3.4 and Lemma 3.5 apply.

In Region II, $|x-y| > a_+t$, so that from (4.5) we see that Γ_+ crosses the real axis at a positive point. We also have in Region II, as noted above, that Γ_+ necessarily intersects Γ_c at some k, say k_+ for Γ_+ and k_c for Γ_c. Thus in Region II we may follow Γ_+ until it intersects Γ_c and from there follow Γ_c to the point at infinity, avoiding the allowed point spectrum. We will denote this new combined contour by Γ_+^c, as our Region II example contour is labeled in Fig. 4.2. We already have estimate (i) along Γ_+, so we need only concern ourselves further with obtaining the estimate along Γ_c. Thus we consider the integral

$$\int_{\Gamma_c^s} e^{\mathrm{Re}(\lambda(k)\,t+\mu_1^+(x-y))}\,\frac{d\lambda}{W_0(\lambda)},$$

where Γ_c^s is the part of Γ_c we must follow in the ball $|\lambda| \leqslant M_l < M_s$. We will make use here of the fact that along Γ_c, $(|d\lambda|/|W_0(\lambda)|) \leqslant C\,|dk|$ in all cases.

It is important to notice that along the dashed-in contour in Fig. 4.2, Γ_0, defined by $\lambda_0(k) = -b_+ k^2 - ia_+ k$, we have $\mathrm{Re}(\mu_1^+)|_{\Gamma_0} = 0$. Further, from (4.3) and (4.5), we see that at any point, P_o, on the outside of this contour we have $\mathrm{Re}(\mu_1^+) < 0$, and at any point, P_i, on the inside of this contour we have $\mathrm{Re}(\mu_1^+) > 0$. In general, $\mathrm{Re}(\mu_1^+)$ becomes more negative as we move away from Γ_0 up and to the right, as can be seen by comparing (4.3) and (4.5).

Recalling that $\lambda_c(k) = -ck^2 - d + ik$, we get, in the region where $\mathrm{Re}(\mu_1^+) < 0$, which includes all of Region II,

$$\mathrm{Re}(-ck^2 t - dt + ikt + \mu_1^+(x-y)) \leqslant -ck^2 t - dt,$$

so that

$$\int_{\Gamma_c^s} e^{\mathrm{Re}(\lambda_c(k)\, t + \mu_1^+(x-y))} \frac{|d\lambda|}{|W_0(\lambda)|} \leqslant C \int_{-\infty}^{+\infty} e^{-ck^2 t} e^{-dt}\, dk = \frac{C}{\sqrt{t}} e^{-dt}.$$

But from our constraint on Region II,

$$|x-y| \leqslant t\sqrt{b_+/c}, \tag{4.13}$$

we get

$$|x-y-a_+ t| \leqslant |x-y| + a_+ t \leqslant t\sqrt{b_+/c} + a_+ t = t(\sqrt{b_+/c} + a_+), \tag{4.14}$$

so that

$$\frac{|x-y-a_+ t|^2}{4tb_+ M} \leqslant \frac{t(\sqrt{b_+/c} + a_+)^2}{4b_+ M} \leqslant td, \tag{4.15}$$

for M sufficiently large. Thus in Region II (and in Regions III and IV, where t is yet more dominant) exponential t-decay is sufficient to give us the claimed estimate.

We finally show that the above estimates persist outside the ball where our small $|\lambda|$ estimate holds. In this case, since $M_s > M_l$, our large $|\lambda|$ estimates apply. We get, with Γ_c^l denoting the part of Γ_c on which Lemma 3.4 fails to hold,

$$\int_{\Gamma_c^l} \mathbf{O}(|\lambda|^{-1/2})\, e^{\mathrm{Re}(\lambda t)} e^{-\mathrm{Re}\sqrt{\lambda/b_s}\,|x-y|/2} |d\lambda| \leqslant \int_{-\infty}^{+\infty} e^{-ck^2 t} e^{-td} dk \leqslant \frac{C}{\sqrt{t}} e^{-td},$$

which leads to the claimed estimate exactly as above.

Case (i), *Region* III. We now move into Region III by taking, for any $0<\varepsilon<d$, the parabolic contour,

$$\lambda_\varepsilon(k)=-b_+k^2-ik\left(2\frac{\varepsilon-d}{a_+}b_++a_+\right)+(\varepsilon-d)\left[\frac{\varepsilon-d}{a_+^2}b_++\operatorname{sgn}(a_+)\right], \tag{4.16}$$

denoted by Γ_ε. In particular, this contour has been chosen so that $\operatorname{Re}(\mu_1^+)|_{\Gamma_\varepsilon}=+d-\varepsilon/a_+>0$.

We check what relationship between x, y and t gives rise to this Γ_ε. By setting real parts equal, we get $-\alpha_+=(d-\varepsilon)/a_+$, leading to

$$x-y=a_+t-2\frac{d-\varepsilon}{a_+}b_+t. \tag{4.17}$$

Notice that (4.17) is the equation describing the line separating Region III from Region IV in Fig. 4.1.

Until Γ_+ intersects Γ_c the previous analysis (leading to (4.4)) holds, as we are still in the small $|\lambda|$ region and away from all eigenvalues. We then need only notice that on Γ_c we get (for $|\lambda|<M_s$)

$$\int_{\Gamma_c^s}e^{\operatorname{Re}(\lambda t+\mu_1^+(x-y))}\frac{|d\lambda|}{|W_0(\lambda)|}\leqslant C\int_{-\infty}^{+\infty}e^{-ck^2t}e^{-dt}e^{(d-\varepsilon)/|a_+|(x-y)}\,dk,$$

where we have used that $\operatorname{Re}(\mu_1^+)|_{\Gamma_c^s}\leqslant(d-\varepsilon)/|a_+|$, by our observation that $\operatorname{Re}\mu_1^+$ decreases as we move to the right of Γ_ε. In Region III, where $|x-y|<ta_+$, this leads to

$$C\int_{-\infty}^{+\infty}e^{-ck^2t}e^{-dt}e^{(d-\varepsilon)/|a_+|(x-y)}\,dk\leqslant C\int_{-\infty}^{+\infty}e^{-ck^2t}e^{-dt}e^{(d-\varepsilon)t}\,dk=\frac{C}{\sqrt{t}}e^{-\varepsilon t},$$

exponential t-decay, as before. We consequently arrive at the claimed estimate by virtue of (4.14) and (4.15), with ε replacing d. A similar computation leading to t-decay follows on Γ_c^l. Hence, as in Region II, we get (i), but with a possibly large scaling factor M.

Case (i), *Region* IV. At last, we extend our analysis into Region IV. The strategy in Region IV will be to use one particular contour that yielded the estimate in Region III, even when the values of x, y and t would suggest a contour further to the left.

In Region IV we have the inequality

$$|x-y|<t\left(a_+-\frac{2(d-\varepsilon)}{a_+}b_+\right). \tag{4.18}$$

As in Region III, $|x-y| \leqslant ta_+$, so that the estimates made along Γ_c still hold. Thus we need only concern ourselves with the portion of Γ^c_+ before Γ_c is intersected, a finite contour we will denote by Γ^*_+. We have, along Γ^*_+,

$$\begin{aligned}
&\int_{-k_+}^{k_+} e^{\operatorname{Re}(\lambda(k)\, t + \mu_1^+(x-y))}\, dk \\
&\quad = \int_{-k_+}^{k_+} e^{-b_+ k^2 t} e^{(\varepsilon - d)[(\varepsilon - d)/a_+^2\, b_+ + \operatorname{sgn}(a_+)]\, t + (d-\varepsilon)/a_+\, (x-y)}\, dk \\
&\quad \leqslant \int_{-k_+}^{k_+} e^{-b_+ k^2 t} e^{b_+ t((\varepsilon - d)^2/a_+^2) + (\varepsilon - d)\, t} e^{(d-\varepsilon)/a_+\, (x-y)} \\
&\quad \leqslant \frac{C}{\sqrt{tb_+}}\, e^{b_+ t((\varepsilon - d)^2/a_+^2) + (\varepsilon - d)\, t + (d-\varepsilon)/a_+\, t(a_+ - 2(d-\varepsilon)/a_+\, b_+)} \\
&\quad = \frac{C}{\sqrt{tb_+}}\, e^{b_+ t((\varepsilon - d)^2/a_+^2) + (\varepsilon - d)\, t + (d-\varepsilon)\, t - 2(d-\varepsilon)^2/a_+^2\, tb_+} \\
&\quad = \frac{C}{\sqrt{tb_+}}\, e^{-(d-\varepsilon)^2/a_+^2\, tb_+},
\end{aligned}$$

which, having exponential time decay, leads to the result as noted previously.

Case (i), *Derivative Estimates.* We now complete Case (i) by achieving the claimed derivative estimates. We have, for $|\lambda| < M_s$, from Lemma 3.4,

$$\frac{\partial^n}{\partial x^n} G_\lambda(x, y) = \frac{\mathbf{O}(1)}{W_0(\lambda)} (\mu_1^+)^n e^{\mu_1^+(x-y)} + \frac{\mathbf{O}(e^{-\alpha x})}{W_0(\lambda)} e^{\mu_1^+(x-y)}, \tag{4.19}$$

and for $|\lambda| > M_l$, from Lemma 3.5,

$$\frac{\partial^n}{\partial x^n} G_\lambda(x, y) = \mathbf{O}(|\lambda|^{(n-1)/2})\, e^{-\operatorname{Re}(\sqrt{\lambda/b_s}/2)\, |x-y|}. \tag{4.20}$$

Therefore an application of Lebesgue's dominated convergence theorem to Dunford's integral leads to

$$\begin{aligned}\left|\frac{\partial^n}{\partial x^n} G(t, x, y)\right| \leqslant & \frac{1}{2\pi} \int_{\Gamma \cap B(0, M_s)} e^{\operatorname{Re} \lambda t} \frac{\mathbf{O}(1)}{W_0(\lambda)} |\mu_1^+|^n e^{\operatorname{Re} \mu_1^+(x-y)} |d\lambda| \\ & + \frac{1}{2\pi} \int_{\Gamma \cap B(0, M_s)} e^{\lambda t} \frac{\mathbf{O}(e^{-\alpha x})}{W_0(\lambda)} e^{\mu_1^+(x-y)} |d\lambda| \\ & + \frac{1}{2\pi} \int_{\Gamma \backslash B(0, M_s)} e^{\operatorname{Re} \lambda t} \mathbf{O}(|\lambda|^{(n-1)/2}) e^{-\operatorname{Re} \sqrt{\lambda/b_s}/2 \, |x-y|} |d\lambda|.\end{aligned} \tag{4.21}$$

We see from (4.21) that there are three new elements introduced into the above analysis, the term $(\mu_1^+)^n$ in the first integral, the $\mathbf{O}(e^{-\alpha x})$ in the second and the term $\mathbf{O}(|\lambda|^{(n-1)/2})$ in the third.

Again we must consider each of our four regions of analysis in Fig. 4.1. In Region I, (4.20) always holds so that we will follow the large $|\lambda|$ analysis of Case (i) with $\mathbf{O}(|\lambda|^{(n-1)/2})$ replacing $\mathbf{O}(|\lambda|^{-1/2})$. We arrive, through a computation similar to (4.10) and (4.11), at

$$\begin{aligned}\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right| & \leqslant \frac{1}{2\pi} \int_{\Gamma_{\lambda_0}} e^{\operatorname{Re} \lambda t} \mathbf{O}(|\lambda|^{(n-1)/2}) e^{-\operatorname{Re} \sqrt{\lambda/b_s}/2 \, |x-y|} |d\lambda| \\ & \leqslant C e^{-|x-y|^2/16 t b_s} \int_{-\infty}^{+\infty} [b_s(\lambda_0 + k^2)]^{(n-1)/2} \\ & \quad \times |-2k b_s + 2i b_s \sqrt{\lambda_0}| \, e^{-b_s k^2 t} \, dk \\ & \leqslant C e^{-|x-y|^2/16 t b_s} \int_{-\infty}^{+\infty} (|\lambda_0|^{n/2} + |k|^n) \, e^{-b_s k^2 t} \, dk \\ & \leqslant C e^{-|x-y|^2/16 t b_s} \left[\frac{|\lambda_0|^{n/2}}{\sqrt{b_s t}} + \frac{1}{(b_s t)^{(n+1)/2}}\right].\end{aligned} \tag{4.22}$$

We now take advantage of the fact that for any n there exists some constant C_n, depending only on n, such that $x^{n/2} e^{-x} \leqslant C_n e^{-x/2}$, to arrive at a bound by

$$C e^{-|x-y|^2/16 t b_s} \left[\frac{(|\lambda_0| \, b_s t)^{n/2}}{(b_s t)^{(n+1)/2}} + \frac{1}{(b_s t)^{(n+1)/2}}\right] \leqslant \frac{C e^{-|x-y|^2/32 t b_s}}{(t b_s)^{(n+1)/2}},$$

which, as in (4.12), is equivalent to the claimed estimate.

Derivative Estimates, Region II. In Region II we follow Γ_+^c and consequently arrive in the same manner as in the Region II analysis above at the estimate

$$\left|\frac{\partial^n}{\partial x^n} G(t, x, y)\right| \leqslant Ce^{-\alpha_+^2 tb_+} \int_{\{k:\, \lambda(k) \in \Gamma_+^*\}} e^{-k^2 b_+ t}(|\mu_1^+(k)|^n + e^{-\alpha |x|})\, dk$$

$$+ Ce^{-dt} \int_{\{k:\, \lambda(k) \in \Gamma_c^s\}} e^{-ck^2 t}(|\mu_1^+(k)|^n + e^{-\alpha |x|})\, dk$$

$$+ Ce^{-dt} \int_{\{k:\, \lambda(k) \in \Gamma_c^l\}} e^{-ck^2 t}\mathbf{O}(|\lambda(k)|^{(n-1)/2})\, dk. \qquad (4.24)$$

In all of these integrals, nothing essential will be lost by extending them over all real k. From (4.3), we have $\mu_1^+|_{\Gamma_+} = ik - \alpha_+$, so that on Γ_+, that is, for the first summand of the first integral above, $|\mu_1^+| \leqslant |k| + |\alpha_+|$, and hence by Young's inequality $|\mu_1^+|^n \leqslant C_n(|k|^n + |\alpha_+|^n)$, for C_n an appropriately large constant dependent only on n. In the latter two integrals, k is bounded away from zero, so that, since $\mu_1^+(k)$ grows at most linearly in k (a consequence of its definition), $|\mu_1^+|^n \leqslant C|k|^n$ and $\mathbf{O}(|\lambda|^{(n-1)/2})\,|d\lambda| \leqslant C|k|^n\, dk$ on these contours for some appropriately large constant C. These two integrals can be combined, then, into a single much simpler integral. Accordingly, we arrive at

$$\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right| \leqslant Ce^{-\alpha_+^2 b_+ t} \int_{-\infty}^{+\infty} e^{-k^2 b_+ t}(|k|^n + |\alpha_+|^n + e^{-\alpha |x|})\, dk$$

$$+ Ce^{-dt} \int_{-\infty}^{+\infty} e^{-ck^2 t}\, |k|^n\, dk.$$

Integrating out k yields a bound by

$$Ce^{-\alpha_+^2 tb_+}\left[\frac{1}{(tb_+)^{(n+1)/2}} + \frac{|\alpha_+|^n}{\sqrt{tb_+}} + \frac{\mathbf{O}(e^{-\alpha x})}{\sqrt{tb_+}}\right] + \left(\frac{Ce^{-td}}{(tb_+)^{(n+1)/2}}\right),$$

where the b_+ has been added in the term from the last integral by appropriately changing the constant, C. The t-decay in the fourth term is, as shown in (4.14) and (4.15), equivalent to the expected decay, the additional algrebraic t-decay coming from the exponential t-decay. In the third term, we note that for $x > y > 0$, $|x| \geqslant |x - y| \geqslant \sqrt{b_+/t}$, so that we have exponential time decay, leading again to the derivative estimate of (i). For the second term, we again employ the inequality, $x^{n/2}e^{-x} < C_n e^{-x/2}$ for some C_n and for all $x > 0$, to arrive at

$$\frac{C(\alpha_+^2 tb_+)^{n/2}\, e^{-\alpha_+^2 tb_+}}{(tb_+)^{(n+1)/2}} \leqslant CC_n \frac{e^{-\alpha_+^2 tb_+/2}}{(tb_+)^{(n+1)/2}}. \qquad (4.25)$$

Finally, we note that the first term of the expression is already in the appropriate form.

Derivative Estimates, *Region* III. A similar analysis will also be appropriate in Region III, using Γ_ε as before. In this region, we again take Γ^c_+, for $|x-y| \geqslant a_+ t$, but such that $\mathrm{Re}(\mu_1^+)|_{\Gamma_\varepsilon} = (d-\varepsilon)/a_+ > 0$. As before, we bound the nth x-derivative of G by the sum of integrals over three domains, Γ^*_+, Γ^s_c, and Γ^l_c. The analysis along Γ^*_+ and Γ^l_c follows almost exactly as before. Along Γ^s_c, we have a partial bound on $|(\partial^n/\partial x^n) G(t, x; y)|$ by

$$
\begin{aligned}
&Ce^{-td}e^{(d-\varepsilon)/a_+\,|x-y|}\int_{\{k:\,\lambda(k)\in\Gamma^s_c\}} e^{-ck^2t}(|k|^n+|\alpha_+|^n+e^{-\alpha x})\,dk\\
&\quad\leqslant Ce^{-td+t(d-\varepsilon)}\left[\frac{1}{(ct)^{(n+1)/2}}+\frac{|\alpha_+|^n}{\sqrt{ct}}+\frac{e^{-\alpha x}}{\sqrt{ct}}\right]\\
&\quad= Ce^{-t\varepsilon}\left[\frac{1}{(ct)^{(n+1)/2}}+\frac{|\alpha_+|^n}{\sqrt{ct}}+\frac{e^{-\alpha x}}{\sqrt{ct}}\right].
\end{aligned}
$$

Through a computation similar to (4.14) and (4.15) and the derivative estimates of Region II we arrive at the claimed estimate.

Derivative Estimates, *Region* IV. In Region IV we remain on a Region III contour, denoting it now by Γ^c_ε. Of the three domains of integration previously considered, only the analysis along $\Gamma^*_\varepsilon\,(:=\Gamma^c_\varepsilon\backslash\Gamma_c)$ gives rise to essentially new analysis. We notice that in Region IV there exists some constant, say $\gamma > 0$, so that $|\alpha_+| > \gamma$ in the entire region. Thus along the fixed (independent of x, y or t) contour Γ^*_ε, $|\mu_1^+| \leqslant C(|k|+1) \leqslant C(|k| + C_1\,|\alpha_+|)$, for some constants C and C_1. This allows the computation

$$
\begin{aligned}
&Ce^{-(d-\varepsilon)^2|a_+|^2\,tb_+}\int_{\{k:\,\lambda(k)\in\Gamma^*_\varepsilon\}} e^{-b_+k^2t}(|\mu_1^+|^n+e^{-\alpha x})\,dk\\
&\quad\leqslant Ce^{-(d-\varepsilon)^2|a_+|^2\,tb_+}\int_{\{k:\,\lambda(k)\in\Gamma^*_\varepsilon\}} e^{-b_+k^2t}(|k|^n+|\alpha_+|^n+e^{-\alpha x})\,dk,
\end{aligned}
$$

which leads to the claimed estimate similarly to (4.14), (4.15), and (4.25). With this computation we have completed our analysis of the first case of Theorem 1.1.

In the remainder of the proof of Theorem 1.1 we will avoid repeating computations whenever possible by referring back to those carried out in Case (i).

Case (ii), *Derivative Estimates.* The only fundamental difference between the anlaysis of Case (i) and the analysis of Case (ii) is in the derivative estimates. In Case (ii) we have less algebraic time decay, a phenomenon consistent with known exact solutions.

Our proof in this case follows exactly as before except that with $y > x > 0$, exponential x-decay does not give rise to exponential time decay as in the case when $x > y > 0$. Thus the analysis of Case (i) remains valid, except where this exponential t-decay was used to obtain the additional algebraic t-decay. Hence, the weaker result.

Case (iii), *Region* I. In Case (iii) we have $x > 0 > y$ with $a_+ > 0$, $a_- < 0$. As before, in Region I the large $|\lambda|$ estimates of Lemma 3.5 hold. The same analysis as given in Case (i) ((4.10) and (4.11)) leads to the estimate

$$|G(t, x; y)| \leqslant C \frac{e^{-|x-y|^2/16tb_s}}{\sqrt{tb_s}}.$$

The difference in this case is that we actually expect what appears to be more, additional exponential y-decay. In fact, in Region I, where $|x - y| \geqslant t/\sqrt{b_+/c}$, this exponential y-decay results from the above decay as illustrated in the computation

$$|G(t, x; y)| \leqslant \frac{C}{\sqrt{tb_s}} e^{-|x-y|^2/16tb_s} = \frac{C}{\sqrt{tb_s}} e^{-|x-y|^2/32tb_s} e^{-|x-y|^2/32tb_s}, \tag{4.26}$$

where one of the exponents in this last expression can be used as in (4.12) to obtain $-|x - y - a_+ t|^2/tb_+ M$ exponential decay, and the other to get exponential y-decay, through (using $x > 0 > y$)

$$\frac{|x-y|^2}{32tb_s} \geqslant \frac{|x-y|}{32b_s} \sqrt{b_+/c} \geqslant \frac{|y|}{32b_s} \sqrt{b_+/c}. \tag{4.27}$$

Case (iii), *Region* II. In Region II we must again analyze the three integrals over the domains, Γ_+^*, Γ_c^s, and Γ_c^l. We begin with

$$\int_{\Gamma_+^*} \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\lambda t} e^{\mu_1^+ x} e^{-\mu_1^- y} \, d\lambda.$$

We rearrange the terms in the above integrand to get

$$\int_{\Gamma_+^*} \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\lambda t} e^{\mu_1^+ (x-y)} e^{(\mu_1^+ - \mu_1^-) y} \, d\lambda.$$

A difficulty encountered here is that μ_1^- cannot be easily evaluated along Γ_+^*. However, we can see that

$$f(\lambda) := \mu_1^+ - \mu_1^- = \frac{a_+ - \sqrt{a_+^2 + 4b_+ \lambda}}{2b_+} - \frac{a_- - \sqrt{a_-^2 - 4b_- \lambda}}{2b_-}$$

satisfies $f(0) = |a_-|/b_- > 0$, so that by the analyticity of f in a neighborhood of the origin, there exist constants $\eta, \delta > 0$ so that $\mathrm{Re}(f(\lambda)) > \delta$ for $|\lambda| < \eta$. We let Γ_c be close enough to the origin (d small enough) so that any Γ_+^* that strikes the real axis within this η-ball also strikes Γ_c within this η-ball. If, as usual, k_+ represents the value of k where Γ_+ meets Γ_c we get, for Γ_+^c crossing the real axis in this small neighborhood of the origin,

$$\begin{aligned} |G(t, x; y)| &\leqslant Ce^{-\delta |y|} \int_{-k_+}^{+k_+} e^{\mathrm{Re}(\lambda(k)\, t + \mu_1^+(k)(x-y))}\, dk \\ &\quad + C \int_{\{k:\, \lambda(k) \in \Gamma_c^s\}} e^{\mathrm{Re}(\lambda(k)\, t + \mu_1^+(k)\, x - \mu_1^-(k)\, y)}\, dk \\ &\quad + C \int_{\{k:\, \lambda(k) \in \Gamma_c^l\}} e^{\mathrm{Re}(\lambda(k)\, t - (\sqrt{\lambda(k)/b_s}/2)\, |x-y|)}\, dk. \end{aligned}$$

The integrand of the first term in the above expression is now in the same form as the same term was in Case (i) of this theorem. As for the second integral, we need only note that $\mathrm{Re}(\mu_1^-) \leqslant -|a_-|/2b_-$ (always for Case (iii)) and $\mathrm{Re}(\mu_1^+)|_{\Gamma_c^s} < 0$ (in Region II for Case (iii)) so that

$$\begin{aligned} &C \int_{\{k:\, \lambda(k) \in \Gamma_c^s\}} e^{\mathrm{Re}(\lambda(k)\, t + \mu_1^+(k)\, x - \mu_1^-(k)\, y)}\, dk \\ &\qquad \leqslant Ce^{-|a_-|/b_-\, |y|} \int_{-\infty}^{+\infty} e^{-ck^2 t - dt}\, dk, \end{aligned}$$

from which the claimed decay follows as in (4.14) and (4.15). The third integral is almost exactly the same as the large $|\lambda|$ integral along Γ_c for Case (i). We need only note that in order to obtain the additional y-decay, we may take advantage of the inequality in this case, $|x-y| \geqslant |y|$, and the fact that there exists some $\gamma > 0$ such that $\mathrm{Re}(\sqrt{\lambda/b_s}/2)|_{\Gamma_c^l} \geqslant \gamma$.

What now remains in Region II is to obtain the result for the gap between our η-ball and the large $|\lambda|$ estimates. We will accomplish this by using one particular contour within the η-ball even as our values of x, y and t would have us take another, further to the right (crossing the positive real axis at a greater value). That is, for a particular $\eta > 0$ we will use the contour, denoted Γ_η,

$$\lambda_\eta(k) = -b_+ k^2 + b_+ \eta^2 + a_+ \eta + ik(-a_+ - 2b_+ \eta),$$

that is, the contour satisfying $\operatorname{Re} \mu_1^+ = -\eta$. We have, then, in this *middle region*, $\alpha_+ \geqslant \eta$ so that $(x - y - a_+ t)/2b_+ t \geqslant \eta$, which gives $|x - y| \geqslant t(2b_+ \eta + a_+)$. Thus on Γ_η^*, the portion of Γ_η before Γ_η intersects Γ_c, we have the bound

$$\begin{aligned}
&\int_{-k_+}^{k_+} e^{-b_+ k^2 t + b_+ \eta^2 t + a_+ \eta t - \eta(x - y)}\, dk \\
&\quad \leqslant \int_{-k_+}^{k_+} e^{-b_+ k^2 t + b_+ \eta^2 t + a_+ \eta t - \eta t(2b_+ \eta + a_+)}\, dk \\
&\quad = \int_{-k_+}^{k_+} e^{-b_+ k^2 t + b_+ \eta^2 t + a_+ \eta t - 2\eta^2 t b_+ - \eta t a_+}\, dk \\
&\quad = e^{-\eta^2 t b_+} \int_{-k_+}^{k_+} e^{-b_+ k^2 t}\, dk \\
&\quad \leqslant \frac{C e^{-\eta^2 t b_+}}{\sqrt{t b_+}}.
\end{aligned}$$

This exponential t-decay leads, as always, (in all regions except Region I) to the claimed decay. We note finally that as our contours move to the right our estimates along Γ_c continue to hold as before, finishing off the analysis for Region II.

Since in Region III we employed in Case (i) a contour arbitrarily close to $|\lambda| = 0$, nothing essential changes in the Case (iii) analysis of Region III from that of Case (i). The same analysis holds for Region IV also, where we simply remain on a contour appropriate for Region III. What is now left is to obtain the claimed derivative estimates for this case.

Case (iii), *Derivative Estimates.* In Region I we obtain, as in (4.22) and (4.23),

$$\left| \frac{\partial^n G(t, x; y)}{\partial x^n} \right| \leqslant C \frac{e^{-|x - y|^2/16 t b_+}}{(t b_+)^{(n+1)/2}}.$$

As in (4.26) and (4.27), by breaking this exponent into two equivalent pieces, we can obtain the the claimed estimate.

In Region II we will again commence our analysis in the ball $|\lambda| \leqslant \eta$. Here, we get from Lemma 3.4 and the above analysis,

$$\left|\frac{\partial^n G(t, x; y)}{\partial x^n}\right| \leqslant Ce^{-\delta |y|} \int_{-k_+}^{+k_+} (|\mu_1^+(k)|^n + e^{-\alpha |x|})\, e^{\mathrm{Re}(\lambda(k)\, t + \mu_1^+(k)(x-y))}\, dk$$

$$+ Ce^{-|a_-|/b_- |y|} e^{-dt} \int_{\{k:\, \lambda(k) \in \Gamma_c^s\}} e^{-ck^2 t}(|\mu_1^+(k)|^n + e^{-\alpha |x|})\, dk$$

$$+ C \int_{\{k:\, \lambda(k) \in \Gamma_c^l\}} |k|^n\, e^{-ck^2 - dt} e^{(\sqrt{\lambda(k)/b_s}/2)\, |x-y|}\, dk.$$

Each term on the right of this inequality is exactly the same as a term dealt with in the first case of this theorem except that we see the additional exponential y-decay appearing. In the third integral this y-decay comes, as before, from the exponent $(\sqrt{\lambda(k)/b_s}/2)\, |x-y|$. The one difference in the analysis here, is that where in Case (i), with $x > y > 0$, exponential x-decay led to exponential t-decay, now, since we have in the $\mathbf{O}(e^{-\alpha |x|})$ terms both exponential x-decay and exponential y-decay, we obtain exponential $|x-y|$-decay, which is equivalent to exponential t-decay. Thus we again achieve the higher order algebraic t-decay as a trivial consequence.

We now move to the right, through Region II, by noting that outside this η-ball we have the relationship $\alpha_+ \geqslant \eta$ so that, on Γ_η^*, we have a bound by

$$Ce^{-\delta |y|} e^{-\eta^2 t b_+} \int_{-\infty}^{+\infty} (|\mu_1^+|^n + e^{-\alpha x})\, e^{-b_+ k^2 t}\, dk$$

$$\leqslant C' e^{-\eta^2 t b_+} \int_{-\infty}^{+\infty} (|\alpha_+|^n + e^{-\alpha x})\, e^{-b_+ k^2 t}\, dk,$$

which we have seen, as in (4.25), leads to the claimed decay. Again nothing essential changes in Regions III and IV from the analysis of Case (i) above, as long as our Region III contour, Γ_ε, lies inside the $|\lambda| \leqslant \eta$ ball until Γ_ε intersects Γ_c.

Case (iv). In the fourth case we have $a_+ < 0$, $a_- > 0$, with $x > 0 > y$. As before the large $|\lambda|$ estimates of Lemma 3.5 lead to the result in Region I.

Over domains in which we have the the small $|\lambda|$ estimates, we consider the integral

$$\frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \frac{\mathbf{O}(1)}{W_0(\lambda)}\, e^{\mu_1^+ x} e^{-\mu_1^- y}\, d\lambda,$$

which can be rewritten as

$$\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\,\frac{\mathbf{O}(1)}{W_0(\lambda)}\,e^{\mu_1^-(x-y)}e^{(\mu_1^+-\mu_1^-)x}\,d\lambda.$$

Note that we have taken a new tack here than in Case (iii), as we now expect additional exponential x-decay rather than additional exponential y-decay. In this case we take the contour Γ_-, a contour defined by

$$\lambda_-(k) := -b_-(k+i\alpha_-)^2 - a_-(k+i\alpha_-), \tag{4.28}$$

where $\alpha_- := (x-y-a_-t)/2b_-t$. Following Γ_-^c rather than Γ_+^c leads to the claimed estimate exactly as in Case (iii), except with x and y switched—the expected result since the two cases are adjoints of one another.

Case (iv), *Derivatives.* The derivative estimate for Case (iv) contains less algebraic t-decay than Case (iii). We can easily gain intuition about this effect by considering the estimate on G in Case (iv) to be an equality. In this case, the computation of derivatives through the product rule would always leave a term with $1/\sqrt{t}$ algebraic decay. We see this arise in the proof because in Case (iv) we no longer have exponential y-decay, so we cannot make the argument of Case (iii) in which a combination of exponential x-decay and exponential y-decay led to exponential t-decay. Otherwise, the proof of Case (iv) follows as in Case (iii), except that we take the contour Γ_-^c rather than Γ_+^c.

Case (v), *Region* I. The analysis of the fifth case of Theorem 1.1 is more subtle than the others, as it is the only case for which significant mass crosses the origin. As mentioned in the introduction we now expect the bulk of this mass to move at a different speed on either side of the origin and thus for our peak to appear when $x = (a_+/a_-)\,y + a_+t$. For this analysis we will use in lieu of Fig. 4.1, Fig. 4.3.

In Region I, we now have the inequality

$$|x-(a_+/a_-)\,y| \geqslant t\sqrt{b_+/c}. \tag{4.29}$$

We note that for $x, -y \geqslant 0$ and $a_+, a_- > 0$, we have $C_1\,|x-y| \geqslant |x-(a_+/a_-)\,y|$, for some constant, C_1, depending only on a_+ and a_-. We can thus again take c small enough so that the contour Γ_{λ_0} lies entirely outside the M_l-ball. This leads in the usual manner to the estimate

$$|G(t,x;y)| \leqslant \frac{C}{\sqrt{b_s t}}\,e^{-|x-y|^2/16tb_s}.$$

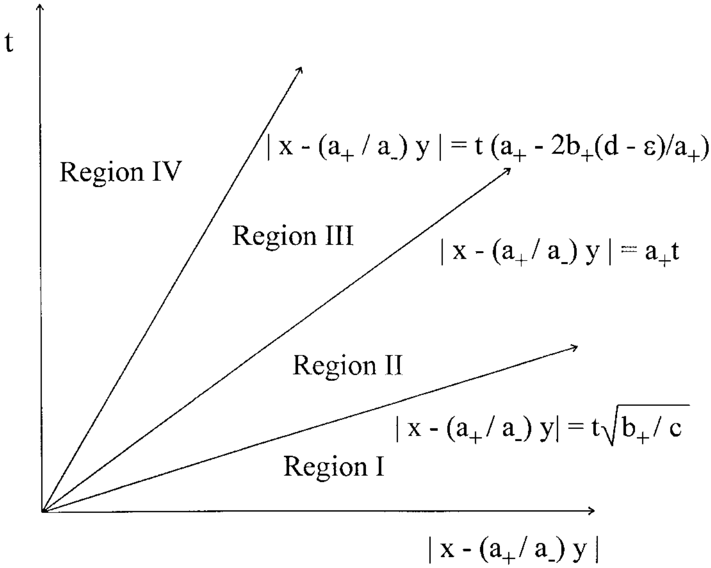

FIG. 4.3. Regions I–IV.

Thus, we have

$$|G(t, x; y)| \leqslant \frac{C}{\sqrt{b_s t}} e^{-|x-(a_+/a_-)y|^2/16tb_s C_1} \leqslant \frac{C}{\sqrt{b_s t}} e^{-|x-(a_+/a_-)y-a_+t|^2/16tb_s M}$$

for some constant M appropriately chosen through a computation similar to (4.12). By changing the constants C_1 and M if necessary, we obtain (v).

Case (v), *Region* II. In Region II, in the domain in which the small $|\lambda|$ estimates hold, we have the integral

$$\int_{\Gamma} \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mathrm{Re}(\lambda t + \mu_1^+ x - \mu_1^- y)}\, d\lambda. \tag{4.30}$$

In this case, we find an appropriate contour through use of the following observation, given as a lemma.

LEMMA 4.1. *For any point* $P \in \mathbb{R}$ *satisfying* $P > \max(-a_-^2/(4b_-), -a_+^2/(4b_+))$, *there exists a contour, say* Γ_P, *so that either* $\mathrm{Re}(\mu_1^+)$ *or* $\mathrm{Re}(\mu_1^-)$ *is constant on the length of* Γ_P *and such that*

$$\mathrm{Re}(\mu_1^+ x - \mu_1^- y)|_{\Gamma_P} \leqslant \mathrm{Re}(\mu_1^+(P)x - \mu_1^-(P)\, y).$$

Proof. Given any such point P there exists exactly one contour passing through P such that $\mathrm{Re}(\mu_1^+) = constant$ along that contour and exactly one contour passing through P such that $\mathrm{Re}(\mu_1^-) = constant$ along that contour. If $\mathrm{Re}(\mu_1^\pm) = -C_\pm$, then these contours will have the form

$$\mathrm{Re}(\lambda) = -b_\pm \frac{1}{(2b_\pm C_\pm + a_\pm)^2} \mathrm{Im}(\lambda)^2 + b_\pm C_\pm^2 + a_\pm C_\pm. \tag{4.31}$$

Of these two, let Γ_P denote the contour that lies farthest to the right. If the two contours lie one atop the other then either contour may be chosen as Γ_P. For definiteness suppose this right-most contour is the contour for which $\mathrm{Re}(\mu_1^+) = constant$ (see Fig. 4.4). By symmetry, the analysis would be the same if the $Re(\mu_1^-) = constant$ contour were the appropriate one. Notice that from (4.31), we can see that as $C_\pm$ increases, the two contours move to the right and open more rapidly. Thus, we have that $\mathrm{Re}(\mu_1^-)|_{\Gamma_P} \leqslant \mathrm{Re}(\mu_1^-(P))$. Since $\mathrm{Re}(\mu_1^+)|_{\Gamma_P} = \mathrm{Re}(\mu_1^+(P))$, this gives the result. ∎

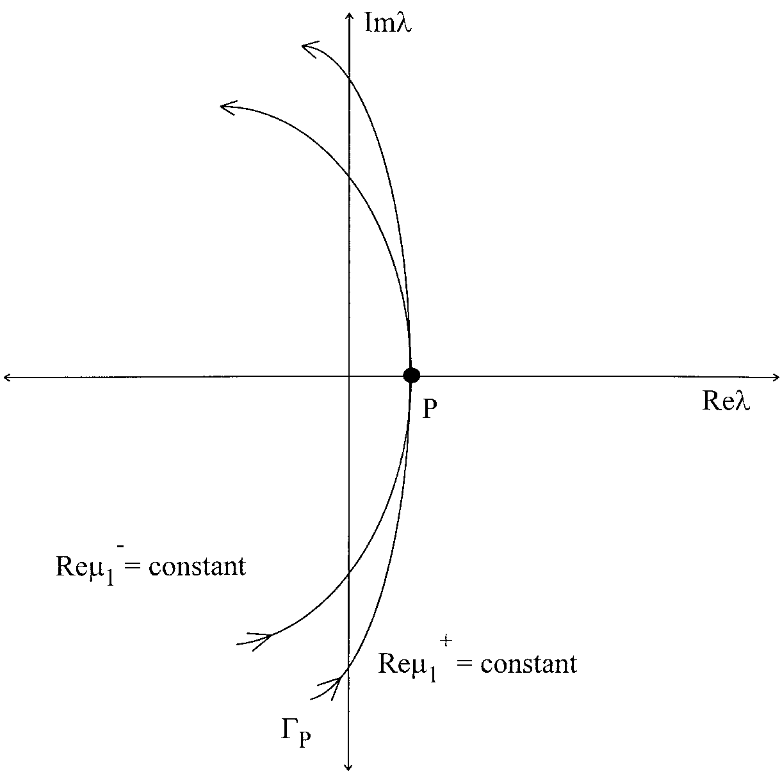

FIG. 4.4. The contours of Lemma 4.1.

Using Lemma 4.1 our approach will be to obtain the claimed estimate in a neighborhood of the origin by finding the optimal point, P, through which to extend a contour. Thus, we want the value of P in the region outlined in Lemma 4.1 for which

$$f(P) := Pt + \mu_1^+(P)\, x - \mu_1^-(P)\, y$$

$$= Pt + \frac{a_+ - \sqrt{a_+^2 + 4Pb_+}}{2b_+}\, x - \frac{a_- - \sqrt{a_-^2 + 4Pb_-}}{2b_-}\, y$$

is minimized. Since we are only concerned with a sufficiently small ball around the origin, we will Taylor expand f around $P=0$ in order to arrive at a more tractable (though approximate) function to minimize. This leads to

$$f(P) = \left[t - \frac{x}{a_+} + \frac{y}{a_-} \right] P + \left[\frac{2b_+ x}{a_+^3} - \frac{2b_- y}{a_-^3} \right] P^2 + \mathbf{O}(P^3).$$

We find the value of P that minimizes the first two terms of $f(P)$ to be

$$P = -\left(t - \frac{x}{a_+} + \frac{y}{a_-} \right) \Big/ \left(\frac{4b_+ x}{a_+^3} - \frac{4b_- y}{a_-^3} \right).$$

In particular, as expected, $\operatorname{sgn}(P) = \operatorname{sgn}(\bar{\alpha}_+)$ and $P = 0$ iff $\bar{\alpha}_+ := (x - (a_+/a_-)\, y - a_+ t)/2b_+ t = 0$.

Again, for definiteness, we will assume that the contour for which $\operatorname{Re}(\mu_1^+) = \textit{constant}$ lies farthest to the right. In our small neighborhood of the origin, where this constant is necessarily small, we see from (4.31) that this is the same as assuming

$$b_+/a_+^2 < b_-/a_-^2 \tag{4.32}$$

(and again if $b_+/a_+^2 = b_-/a_-^2$, either contour will suffice). A nice intuitive way of looking at this inequality is in the case when $b_+ = b_-$, where it simply observes that $a_+ > a_-$, that is, that the mass convects more rapidly to the right of the origin than to the left.

If we let $\beta_+ := -(a_+ - \sqrt{a_+^2 + 4Pb_+})/2b_+$, then we will be taking the contour given by

$$\lambda_P(k) := -b_+(k + i\beta_+)^2 - ia_+(k + i\beta_+),$$

which satisfies $\operatorname{Re} \mu_1^+ = -\beta_+$. We now need only show that for P sufficiently small, that is, in a neighborhood of $\bar{\alpha}_+ = 0$, this contour leads to the claimed estimate. Following our previous notation, we will let Γ_P^c

denote the contour constructed by following Γ_P until it intersects with Γ_c and then following Γ_c out to the point at infinity. We further define $\Gamma_P^* := \Gamma_P^c \backslash \Gamma_c$ and note that along Γ_P in Region II, $\mathrm{Re}(\mu_1^+)$, $\mathrm{Re}(\mu_1^-) < 0$, as before. Thus from (4.30) and Lemma 4.1 we have (in Region II)

$$|G(t, x; y)| \leqslant C \int_{\{k:\, \lambda(k) \in \Gamma_P^*\}} \times e^{-b_+ k^2 t + b_+ \beta_+^2 t + a_+ \beta_+ t - \beta_+ x - ((a_- - \sqrt{a_-^2 + 4Pb_-})/2b_-)\, y}\, dk + \frac{C}{\sqrt{t}} e^{-td},$$

where the second term follows from the previous analysis along Γ_c. The first term is bounded by

$$\frac{C}{\sqrt{b_+ t}} e^{b_+ \beta_+^2 t + a_+ \beta_+ t - \beta_+ x - ((a_- - \sqrt{a_-^2 + 4Pb_-})/2b_-)\, y}. \tag{4.33}$$

Thus, in order to obtain the estimate in a sufficiently small ball around the origin, we need only show that the exponent of (4.33) leads to the appropriate decay. In order to see this, let $h(P)$ denote the argument of the exponential divided by t. That is, let

$$h(P) := b_+ \beta_+^2(P) + a_+ \beta_+(P) - \beta_+(P)\bar{x}(P) - \frac{a_- - \sqrt{a_-^2 + 4Pb_-}}{2b_-}\, \bar{y},$$

where $\bar{x} = x/t$ and $\bar{y} = y/t$. Note that we will think of $\bar{y}$ as fixed so that $\bar{x}$ will depend only on P. An expression for $\bar{x}(P)$ can easily be obtained by inverting the expression for $P(x)$ above, giving

$$\bar{x}(P) = \left(1 + \frac{\bar{y}}{a_-} - \frac{4Pb_- \bar{y}}{a_-^3}\right) \Big/ \left(\frac{1}{a_+} - \frac{4Pb_+}{a_+^3}\right).$$

An important remark at this point is that both $\bar{x}$ and $\bar{y}$ are bounded in this region because α_+ is bounded, so there exists some constant, say M_1, so that $\bar{\alpha}_+ = x - (a_+/a_-)\, y - a_+ t/2b_+ t \leqslant M_1$, which gives that $\bar{x} - (a_+/a_-)\, \bar{y} \leqslant 2b_+ M_1 - a_+$, a relationship that, with x and $-y$ both positive, gives a bound on both $\bar{x}$ and $\bar{y}$.

We next note that for $\bar{y}$ fixed, $h(P)$ satisfies the following derivative conditions:

$$h(0) = 0, \qquad h'(0) = 0, \quad h''(0) = -\frac{6b_+}{a_+^2} + \frac{6\bar{y}}{a_-}\left[\frac{b_-}{a_-^2} - \frac{b_+}{a_+^2}\right].$$

Since we are in the case $b_+/a_+^2 < b_-/a_-^2$ and $\bar{y} < 0$, we have $h''(0) \leqslant -6b_+/a_+^2$ and thus a *strict* local maximum of $h(P)$ at $P = 0$. Consequently, in a sufficiently small neighborhood of the origin (say $|\beta_+| \leqslant \eta$, $\bar{\alpha}_+ \leqslant \delta$) $h(P) = 0$ iff $P = 0$ iff $\bar{\alpha}_+ = 0$, and, moreover, $h(P) < 0$ in a neighborhood of $P = 0$, so that there exists a sufficiently large constant M such that $h(P) \leqslant -\bar{\alpha}_+^2 b_+/M$ in that neighborhood. It should be remarked that this last inequality is valid because there is no t dependence involved, and because $\bar{x}$ and $\bar{y}$ are both bounded so that an M can be chosen independently of $\bar{x}$ and $\bar{y}$. Substituting this estimate into (4.33) gives the claimed estimate in a a sufficiently small ball around $\bar{\alpha}_+ = 0$.

Next, we extend this estimate to the remainder of $\bar{\alpha}_+ \geqslant 0$ between our small and large $\bar{\alpha}_+$ estimates by remaining on a fixed contour passing through a point, P_η, which satisfies $|\beta_+(P_\eta)| \leqslant \eta$, even as our values of x, y and t would suggest contours farther to the right. We will denote this contour by Γ_η and define it through

$$\lambda_\eta := -b_+(k + i\eta)^2 - ia_+(k + i\eta).$$

We now employ the relationship in this region, $P > P_\eta$, which gives

$$x \geqslant \frac{-4P_\eta b_- y a_-^{-3} + y a_-^{-1} + t}{-4P_\eta b_+ a_+^{-3} + a_+^{-1}},$$

an expression positive for P_η sufficiently small, since $|y|/a_- \leqslant t$ is necessary for the kernel to have crossed the origin. As before we have (4.33), where the second integral is treated as usual and the first becomes bounded by

$$\frac{C}{\sqrt{b_+ t}} e^{b_+ \eta^2 t + a_+ \eta t - \eta((-4P_\eta b_- y a_-^{-3} + y a_-^{-1} + t)/(-4P_\eta b_+ a_+^{-3} + a_+^{-1})) - ((a_- - \sqrt{a_-^2 + 4P_\eta b_-})/2b_-)\, y}.$$

The point P_η was chosen so that the above exponent is strictly less than zero. Therefore on the bounded region $\delta \leqslant \bar{\alpha}_+ \leqslant (\sqrt{b_+/c} - a_+)/(2b_+)$, the bound on $\bar{\alpha}_+$ between Region I and Region II, we get the same bound as before, by compactness.

Case (v), *Regions* III *and* IV. We now carry the analysis into Regions III and IV by noting that the Region III analysis only occurs in a sufficiently small ball around the origin and consequently follows from the analysis of Region II. Extension to Region IV is carried out as before by using a contour appropriate in Region III, even as our values of x, y and t would suggest we employ a contour farther to the left. The analysis then follows precisely as in the Region II analysis of this case, except that now $P<0$ and $\eta<0$, and we have x bounded above.

Case (v), *Derivative Estimates.* The claimed estimates on the derivatives of $G(t, x; y)$ are obtained as before in the large $|\lambda|$ region (Region I).

Again, for Region II we consider a sufficiently small ball around the origin ($\beta_+ \leqslant \eta$). In such a ball, we have by virtue of the previous analysis

$$\left|\frac{\partial^n}{\partial x^n} G(t, x; y)\right|$$

$$\leqslant C \int_{-k_+}^{+k_+} (|\mu_1^+|^n + e^{-\alpha|x|})$$

$$\times e^{-b_+ k^2 t + b_+ \beta_+^2 t + a_+ \beta_+ t - \beta_+ x - ((a_- - \sqrt{a_-^2 + 4Pb_-})/2b_-) y}\, dk$$

$$+ C \int_{-\infty}^{+\infty} |k|^n e^{-ck^2 t - dt}\, dk.$$

On Γ_P, $\mathrm{Re}(\mu_1^+) = (a_+ - \sqrt{a_+^2 + 4Pb_+})/2b_+$, which is zero when $P=0$ ($\bar{\alpha}_+ = 0$), so that for some constant C we have $|\mathrm{Re}(\mu_1^+)| \leqslant C|\bar{\alpha}_+|$ for P (and thus $\bar{\alpha}_+$) bounded. Also, since $\mathrm{Im}(\mu_1^+) = 0$ if $k=0$, we have $\mathrm{Im}(\mu_1^+) \leqslant C|k|$ for some constant C for bounded $|k|$. Hence, $|\mu_1^+| \leqslant C|\bar{\alpha}_+| + C|k|$, and so $|\mu_1^+|^n \leqslant C(|\bar{\alpha}_+|^n + |k|^n)$. This, along with the preceding analysis, gives the result for the first summand of the first integral, with stronger algebraic t-decay than claimed in the statement of Theorem 1.1. For the second summand of the first integral, we achieve only $1/\sqrt{t}$ decay, but with additional exponential x-decay. For the second integral we need only note that with $|k|$ bounded away from zero and $|\mu_1^+|$ having at most growth of rate linear in $|k|$ along Γ_P, there exists a constant C so that $|\mu_1^+| \leqslant C|k|$ for $k \in [-k_c, k_c]^c$ which, as in previous cases, leads to the claimed result. Derivative estimates in Regions III and IV follow similarly.

We note finally that the weaker algebraic time decay stated in Theorem 1.1 for this case occurs (without additional exponential x-decay) in the case $b_+/a_+^2 > b_-/a_-^2$, when there is a build-up of mass at the origin. In the analysis this becomes clear as we take the contour on which $\mathrm{Re}(\mu_1^-) =$ *constant* as Γ_P.

Case (vi). The final case, $a_- < 0$, $a_+ < 0$ with $x > 0 > y$, has exponential $|x|$, $|y|$ and t decay and hence is effectively independent of path. In the case of large $|\lambda|$ this decay is easily seen in the previous manner.

The analysis in Regions II through VI in this case is simplified by the strength of our $|x|$ and $|y|$ decay. In fact, we can content ourselves with taking Γ_c in each of these regions. We get

$$|G(t, x; y)| \leqslant C \int_{\{k:\, \lambda(k) \in \Gamma_c \cap B(0, M_s)\}} e^{-ck^2 - dt} e^{\operatorname{Re}(\mu_1^+ x)} e^{\operatorname{Re}(-\mu_1^- y)}\, dk$$

$$+ C \int_{\{k:\, \lambda(k) \in \Gamma_c \setminus B(0, M_s)\}} e^{-ck^2 - dt}\, dk.$$

Noting that in this case $\operatorname{Re}(\mu_1^+) < 0$ and $\operatorname{Re}(-\mu_1^-) > 0$ everywhere, we get in all regions

$$|G(t, x; y)| \leqslant C \int_{-\infty}^{+\infty} e^{-ck^2 - dt}\, dk,$$

which leads in Regions II through IV to the stated result through (4.14) and (4.15), the t-decay giving us decay along any path we choose.

Derivative bounds follow immediately from the previous large $|\lambda|$ analysis and the fact that we have exponential time decay in all other regions, leading trivially to the claimed algebraic time decay.

This completes the proof of Theorem 1.1. ∎

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